Vortex helices for the Gross-Pitaevskii equation

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Abstract : We prove the existence of travelling vortex helices to the Gross-Pitaevskii equation in $\mathbb{R}^3$. These solutions have an infinite energy, are periodic in the direction of the axis of the helix and have a degree one at infinity in the orthogonal direction.

Résumé : Nous prouvons l'existence d'ondes progressives à vorticité sur une hélice pour l'équation de Gross-Pitaevskii dans $\mathbb{R}^3$. Ces solutions sont d'énergie infinie, périodiques dans la direction de l'axe de l'hélice et ont un degré un dans la direction orthogonale.

Keywords : Travelling wave; non linear Schrödinger equation; helix; Ginzburg-Landau; vorticity.

1 Introduction

1.1 Statement of the result

In this paper, we are interested in the existence of travelling waves solutions to the Gross-Pitaevskii equation in space dimension 3

$$i\frac{\partial \psi}{\partial t} + \Delta\psi + (1-|\psi|^2)\psi = 0, \tag{1}$$

where $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$. This equation is used as a model for Bose-Einstein condensates, nonlinear optics and superfluidity. On a formal level, it possesses two important quantities constant in time

- the energy
$$E(\psi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\psi(.,t)|^2 + \frac{1}{2}(1-|\psi(.,t)|^2)^2 \, dx,$$
- the momentum
$$\vec{P}(\psi) = Im\int_{\mathbb{R}^3} \psi \cdot \overline{\nabla\psi} \, dx = \int_{\mathbb{R}^3} (i\psi, \nabla\psi) \, dx,$$

where $(.,.)$ is the scalar product in $\mathbb{R}^2 \simeq \mathbb{C}$. The first component of $\vec{P}$ is denoted $P(\psi) = \int_{\mathbb{R}^3}(i\psi, \partial_1\psi)\, dx$.

Travelling waves solutions to (1) are solutions of the form (up to a rotation)

$$\psi(x,t) = U(x_1 - Ct, x_2, x_3).$$

The equation on ψ reads now on U

$$iC\frac{\partial U}{\partial x_1} = \Delta U + (1 - |U|^2)U. \tag{2}$$

The question of the existence of such travelling waves for small speeds has been studied in [BS] in dimension 2 and in [BOS] and [C1] in dimension larger than 2. We refer to these papers for details and references about the Gross-Pitaevskii equation. In [BS], travelling waves with a structure of two vortices of degrees 1 and -1 are exhibited, and in [BOS] and [C1] the travelling wave is a vortex ring (like a "smoke ring").

We consider a function U_L^* defined in the following way. We use cylindrical coordinates (x_1, r, θ), where $(r,\theta) \in \mathbb{R}_+ \times (\mathbb{R}/2\pi\mathbb{Z})$ are the polar coordinates in the (x_2, x_3)-plane. We set $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ (we do not identify $\mathbb{T}$ with $\mathbb{S}^1$ to be able to define ∂_{x_1} for example). We fix $L \geq 0$ and define in cylindrical coordinates

$$\mathcal{H}_L := \{x \in \mathbb{T} \times \mathbb{R}^2,\ r = L,\ x_1 = \theta\}.$$

This is an helix of axis x_1, of pitch L, and length $\mathbf{M}(\mathcal{H}_L) = 2\pi\sqrt{1+L^2}$, that we denote $\vec{\mathcal{H}}_L$ when endowed with the orientation given by the natural parametrization

$$\mathbb{T} \ni \theta \mapsto \gamma(\theta) := (\theta, L\cos\theta, L\sin\theta).$$

If $L = 0$, then $\mathcal{H}_0 = \mathbb{T} \times \{0\}$ is the x_1 axis. We may then see $2\pi\vec{\mathcal{H}}_L$ as a prescribed vorticity and consider a map $U_L^* \in \mathcal{C}^\infty(\mathbb{T} \times \mathbb{R}^2 \setminus \mathcal{H}_L, \mathbb{S}^1)$, which will be precisely defined at the end of the subsection, such that its vorticity concentrates on the helix $\mathcal{H}_L$ in the sense that

$$\operatorname{curl}(U_L^* \times \nabla U_L^*) = 2\pi\vec{\mathcal{H}}_L \qquad \text{and} \qquad \operatorname{div}(U_L^* \times \nabla U_L^*) = 0, \tag{3}$$

that is the vector field $U_L^* \times \nabla U_L^*$, representing the gradient of the phase of U_L^*, is given in the figure below. The map U_L^* is therefore smooth outside $\mathcal{H}_L$, is $\mathbb{S}^1$-valued and has a degree one around $\vec{\mathcal{H}}_L$ and

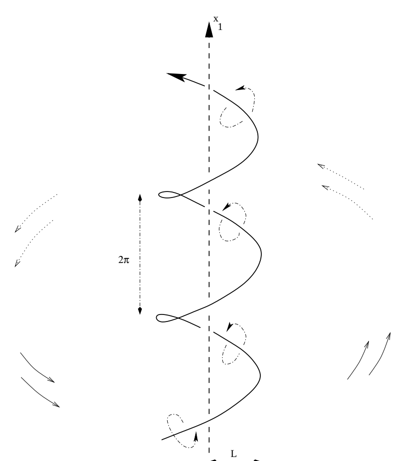

Figure 1: **The vector field $U_L^* \times \nabla U_L^*$**

at infinity (in the (x_2, x_3)-plane). Our main result states the existence, after rescaling, of solutions to (2) close to U_L^*. Due to the degree one at infinity, they are of infinite energy. Moreover, these solutions are periodic in the x_1 variable of the axis of the helix.

Theorem 1. *For every $L > 0$, there exists $\varepsilon_0(L) > 0$ such that, for every $0 < \varepsilon < \varepsilon_0(L)$, there exists a solution U_ε to (2), $\frac{2\pi}{\varepsilon}$-periodic in the x_1 variable, with $C = C(\varepsilon)$ verifying, if $\varepsilon \to 0$,*

$$\frac{C(\varepsilon)}{\varepsilon|\log\varepsilon|} \to \frac{1}{\sqrt{1+L^2}} \qquad \text{and} \qquad P(U_\varepsilon) = 2\Big(\pi\frac{L}{\varepsilon}\Big)^2. \tag{4}$$

Moreover,

$$|U_\varepsilon(x)| \to 1 \qquad \text{as} \quad |(x_2,x_3)| \to +\infty \tag{5}$$

and, for every $k \in \mathbb{N}$,

$$U_\varepsilon\Big(\frac{x}{\varepsilon}\Big) \to U_L^* \qquad \text{in} \quad \mathcal{C}^k_{loc}(\mathbb{T}\times\mathbb{R}^2 \setminus \mathcal{H}_L). \tag{6}$$

Remark 1. In the limiting case $L = 0$, we can find solutions of (2) independent of x_1, that is $U(x) = V(x_2,x_3)$, with V solution of infinite energy (in $\mathbb{R}^2$) and with a degree one at infinity of

$$\Delta V + V(1-|V|^2) = 0 \qquad \text{in } \mathbb{R}^2. \tag{7}$$

These solutions have been studied in [BMR] and also [Sha], [San1] and [Mi]. The associated functions U clearly have a vanishing momentum and are solutions of (2) for any speed $C \in \mathbb{R}$. There exists a particular radially symmetric solution of (7) of degree one at infinity of the form

$$V_0(z) = \rho(|z|)\frac{z}{|z|},$$

where $\rho(r)$ increases from 0 to 1 as r goes from 0 to $+\infty$.

Remark 2. It is important to note that the solution is $\frac{2\pi}{\varepsilon}$-periodic in the x_1 variable, and its singular set is an helix of pitch $\frac{L}{\varepsilon}$. Therefore, we will work with functions U which are defined on $\mathbb{T}_\varepsilon \times \mathbb{R}^2$, with $\mathbb{T}_\varepsilon := \mathbb{R}/(\frac{2\pi}{\varepsilon}\mathbb{Z})$.

Remark 3. We finally emphasize that the momentum in (4) is not exactly the one already introduced. Indeed, since the solution U_ε is periodic in the x_1 variable, the integral which defines the momentum is clearly not convergent in $\mathbb{R}^3$. We will instead consider a momentum defined only on a period, that is $\mathbb{T}_\varepsilon \times \mathbb{R}^2$. Even in this case, we clarify just below the definition.

We clarify the notion of momentum for our problem, and adapt to the situation with a degree one at infinity the definition given in [BOS]. Note that neither the definition of P, since $(iU, \partial_1 U)$ may not be in L^1 at infinity, nor an energy space is clear, since the degree one at infinity makes the energies to diverge. We denote D_R ($R > 0$) the disk in $\mathbb{R}^2$ of radius R centered at 0. We consider the class of functions

$$Y_\varepsilon := \{U \in H^1_{loc} \cap L^\infty(\mathbb{T}_\varepsilon \times \mathbb{R}^2, \mathbb{C}),\ \textstyle\int_{\mathbb{T}_\varepsilon\times\mathbb{R}^2} |\partial_1 U|^2 + \frac{1}{2}(1-|U|^2)^2 < \infty, \exists R > 0 \text{ s.t.}$$

$$\text{for } r \geq R,\ |U(x)| \geq 1/2 \text{ and } U \text{ has degree one outside } \mathbb{T}_\varepsilon \times D_R\}.$$

Note that if $U \in Y_\varepsilon$ and $|U(x)| \geq 1/2$ for $r \geq R$, the degree of U outside $\mathbb{T}_\varepsilon \times D_R$ is well-defined. Indeed, from [BLMN], we know that, for every $R' > R$, since

$$\frac{U}{|U|} \in H^1(\mathbb{T}_\varepsilon \times (D_{R'} \setminus \bar{D}_R), \mathbb{S}^1),$$

then its degree on almost every slice $\{x_1\} \times (D_{R'} \setminus \bar{D}_R)$ is well-defined and is independent of x_1 and $R' > R$: we will call this integer the degree of U outside $\mathbb{T}_\varepsilon \times D_R$. For $U \in Y_\varepsilon$, we may then write

$$U(x) = \rho(x)\exp(i\varphi(x) + i\theta),$$

for $r \geq R$, where $\rho(x) = |U(x)| \geq 1/2$ and $\varphi \in H^1_{loc}(\mathbb{T}_\varepsilon \times (\mathbb{R}^2 \setminus D_R), \mathbb{R})$ is well-defined modulo a multiple of 2π (note that imposing $\partial_1 U \in L^2(\mathbb{T}_\varepsilon \times \mathbb{R}^2)$ prevents U from having a degree in the x_1 variable). We define then

$$P(U) := \int_{\mathbb{T}_\varepsilon \times \mathbb{R}^2} (iU, \partial_1 U)\chi + \int_{\mathbb{T}_\varepsilon \times \mathbb{R}^2} (1-\chi)(\rho^2 - 1)\partial_1 \varphi + \int_{\mathbb{T}_\varepsilon \times \mathbb{R}^2} \varphi \partial_1 (1-\chi), \tag{8}$$

where χ is a smooth function compactly supported, such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $\mathbb{T}_\varepsilon \times D_R$. It is easy to verify that this definition of $P(U)$ does not depend on the exact choice of χ and φ.

For our problem, it is convenient to perform the rescaling

$$u_\varepsilon(x) := U_\varepsilon\Big(\frac{x}{\varepsilon}\Big), \qquad c_\varepsilon := \frac{C(\varepsilon)}{\varepsilon |\log \varepsilon|}.$$

The function u_ε is then defined in $\mathbb{T} \times \mathbb{R}^2$ and equation (2) reads now on u_ε

$$ic_\varepsilon |\log \varepsilon| \frac{\partial u_\varepsilon}{\partial x_1} = \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2). \tag{9}$$

The expressions of the (diverging) energy and momentum are now

$$E_\varepsilon(u_\varepsilon) = \varepsilon E(U_\varepsilon) = \frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}^2} |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \; dx = \int_{\mathbb{T} \times \mathbb{R}^2} e_\varepsilon(u_\varepsilon) \; dx$$

and

$$p(u_\varepsilon) = \varepsilon^2 P(U_\varepsilon) = \int_{\mathbb{T} \times \mathbb{R}^2} (iu_\varepsilon, \partial_1 u_\varepsilon) \; dx.$$

Finally, we would like to mention why we have been interested in these solutions. In [BOS] (see Theorem 4), the study of the asymptotic of a general Ginzburg-Landau equation including (9) in a domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$ under assumption $\sup_\varepsilon |c_\varepsilon| < \infty$ for solutions u_ε satisfying the natural energy bound

$$E_\varepsilon(u_\varepsilon) \leq M_0 |\log \varepsilon|$$

leads to the mean curvature equation for the concentration set

$$\vec{H}(x) = \star\Big(c\vec{e}_1 \wedge \star \frac{dJ_*}{d\mu_*}\Big),$$

where (all these limits are for a subsequence $\varepsilon_n \to 0$) $c = \lim_{\varepsilon \to 0} c_\varepsilon$, $\star$ is Hodge duality, J_* is a limiting measure of the jacobian, μ_* a limiting measure of $\frac{e_\varepsilon(u_\varepsilon)dx}{|\log \varepsilon|}$, $\frac{dJ_*}{d\mu_*}$ is the Radon-Nikodym derivative and $\vec{H}$ is the generalized mean curvature of the varifold $V(\Sigma_{\mu_*}, \Theta_*)$ (Θ_* is the 1-dimensional density of μ_* and $\Sigma_{\mu_*} = \{\Theta_* > 0\}$ its geometrical support). If $N = 3$ and $\frac{d\|J_*\|}{d\mu_*} = 1$, the singular set is a smooth curve γ and this equation rewrites

$$\vec{\kappa} = c\vec{e}_1 \times \vec{\tau}, \tag{10}$$

where $\vec{\tau}$ is the unit tangent and $\vec{\kappa} := \frac{d\vec{\tau}}{ds}$ is the curvature vector of γ. The solutions in $\mathbb{R}^3$ are the circles $a + \{0\} \times \partial D(0, c^{-1})$ ($a \in \mathbb{R}^3$), the straight lines $a + \mathbb{R}\vec{e}_1$ ($a \in \mathbb{R}^3$), and helices of axis parallel to $\vec{e}_1$. The case of a singular circle comes from Theorem 1 in [BOS] (for $N = 3$). A straight line singular set comes from a two dimensional solution (independent of x_1) of the classical Ginzburg-Landau equation in two dimensions, having a singularity of degree 1 in $(x_2, x_3) = (0, 0)$, as the map V_0 (see Remark 1), having radial symmetry. We have constructed the last type of solution.

Definition of the map U_L^*. In order to define precisely the map U_L^*, we note that the natural vector field $\vec{v}$ verifying (3) is given by Biot-Savart law

$$\vec{v}(x) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times(2\pi\vec{\mathcal{H}}_L(y))}{||x-y||^3}\,dy = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{(x-\gamma(\theta))\times\gamma'(\theta)}{||x-\gamma(\theta)||^3}\,d\theta. \tag{11}$$

Note that the integral is convergent since $\|\gamma'\|^2 = 1+L^2$ and $\|\gamma(\theta)\| \sim |\theta|$ for $|\theta| \to +\infty$. By construction, the vector field $\vec{v}$ is smooth outside $\mathcal{H}_L$, satisfies $\operatorname{div}\vec{v} = 0$ and its vorticity

$$\operatorname{curl}\vec{v} = 2\pi\vec{\mathcal{H}}_L$$

is concentrated on $\mathcal{H}_L$. Moreover, $\vec{v}$ has a circulation 2π around $\vec{\mathcal{H}}_L$. We remark that we could have reversed the orientation of the helix, which would have led to the vector field $-\vec{v}$. Another natural helix, turning in the other sense, is

$$\tilde{\mathcal{H}}_L := \{x \in \mathbb{T}\times\mathbb{R}^2, r = L, x_1 = -\theta\},$$

which may be seen as the image of $\mathcal{H}_L$ by the symmetry $x \mapsto (-x_1, x_2, x_3)$, and will be denoted $\vec{\tilde{\mathcal{H}}}_L$ when endowed with the orientation $\mathbb{T} \ni \theta \mapsto \tilde{\gamma}(\theta) := (-\theta, L\cos\theta, L\sin\theta)$. The oriented helices $\vec{\mathcal{H}}_L$ and $\vec{\tilde{\mathcal{H}}}_L$ are "right-hand" for the natural orientation of $\mathbb{T}\times\mathbb{R}^2$. The reference vortex helix U_L^* is defined to be the only map $U_L^* \in \mathcal{C}^\infty(\mathbb{T}\times\mathbb{R}^2\setminus\mathcal{H}_L, \mathbb{S}^1)$ such that

$$U_L^* \times \nabla U_L^* = \vec{v},$$

where, if $a, b \in \mathbb{C} \simeq \mathbb{R}^2$, $a \times b = a_1b_2 - a_2b_1$ denotes the exterior product. The map U_L^* is unique up to a phase change (since $\mathbb{T}\times\mathbb{R}^2\setminus\mathcal{H}_L$ is connected and $U_L^*\times\nabla U_L^*$ is the gradient of the phase of U_L^*). The map U_L^* has therefore a degree one around $\vec{\mathcal{H}}_L$ and at infinity (in the (x_2, x_3)-plane). Changing the orientation of the helix is only complex conjugation for the solution (up to a phase change). Furthermore, changing the helix $\vec{\mathcal{H}}_L$ for $\vec{\tilde{\mathcal{H}}}_L$ changes $\vec{v}$ for $\vec{\tilde{v}}(x) := (-v_1, v_2, v_3)(-x_1, x_2, x_3)$, still of circulation 2π around $\tilde{\mathcal{H}}_L$, and changes U_L^* for $\tilde{U}_L^*(x) := U_L^*(-x_1, x_2, x_3)$, still of degree one at infinity. In the degenerate case where $L = 0$, $\mathcal{H}_0$ is just the axis $\mathbb{T}\times\{0\}$ and U_0^* is then the 2-dimensional map (in the (x_2, x_3)-plane) $U_0^*(x) = \frac{(x_2,x_3)}{|(x_2,x_3)|}$ with a singularity of degree one at 0.

Remark 4. Denoting (U_ε, C, P) as in Theorem 1 and $\tilde{U}_\varepsilon(x) := U_\varepsilon(-x_1, x_2, x_3)$, we remark that $\bar{U}_\varepsilon$ is solution for $(-C, -P)$ and the oriented helix $\mathcal{H}_L$ with reverse orientation; $\tilde{U}_\varepsilon$ (resp. $\bar{\tilde{U}}_\varepsilon$) is solution for $(-C, P)$ (resp. $(C, -P)$) with the helix $\vec{\tilde{\mathcal{H}}}_L$ (resp. with the other orientation).

1.2 Stability of the solution

Concerning the stability of this solution, U_ε must be seen as a minimizer on the whole $\mathbb{T}_\varepsilon\times\mathbb{R}^2$, with the constraints of degree one at infinity and $P = 2\pi^2(\frac{L}{\varepsilon})^2$ but in view of the infinite energy, we can only allow local perturbations, which will preserve the condition of degree one at infinity.

Theorem 2. *For all $0 < \varepsilon < \varepsilon_0(L)$, $U_\varepsilon \in Y_\varepsilon$ and is a constrained minimizer in the following sense. For all $R > 0$, for all $V \in H^1_{loc}\cap L^\infty(\mathbb{T}_\varepsilon\times\mathbb{R}^2, \mathbb{C})$ such that*

$$V = U_\varepsilon \text{ outside } \mathbb{T}_\varepsilon \times D_R$$

(then $V \in Y_\varepsilon$ and $P(V)$ is well-defined) and such that

$$P(V) = 2\Big(\pi\frac{L}{\varepsilon}\Big)^2,$$

then

$$E(V, \mathbb{T}_\varepsilon\times D_R) \geq E(U_\varepsilon, \mathbb{T}_\varepsilon\times D_R).$$

The proof of Theorem 2 is based on a decay result for the energy at infinity (keep in mind that the solution has a degree one at infinity in the variables (x_2, x_3)).

Proposition 1. *There exist smooth maps $\varphi, \rho : \mathbb{T} \times (\mathbb{R}^2 \setminus D_{L+1}) \to \mathbb{R}$ such that for $\varepsilon|(x_2, x_3)| \geq L+1$,*

$$U_\varepsilon(x) = \rho(\varepsilon x)e^{i\varphi(\varepsilon x)+i\theta} = u_\varepsilon(\varepsilon x),$$

and $\rho \geq 1/2$. There exists $C_L > 0$ and $\lambda = \lambda(L) \in (0,1]$ such that, for $r \geq L+1$,

$$\int_{\mathbb{T}\times(\mathbb{R}^2\setminus D_r)} |\nabla\rho|^2 + \rho^2|\nabla\varphi|^2 + \frac{(1-\rho^2)^2}{2\varepsilon^2} \leq \frac{C_L}{r^\lambda}, \tag{12}$$

that is, for $L+1 \leq r \leq R$,

$$\Big| E_\varepsilon(u_\varepsilon, \mathbb{T} \times (D_R \setminus D_r)) - 2\pi^2 \log\Big(\frac{R}{r}\Big)\Big| \leq \frac{C_L}{r^\lambda}.$$

Furthermore, the asymptotic of the energy near the helix as $\varepsilon \to 0$ is

$$\frac{\varepsilon}{\pi|\log\varepsilon|}E(U_\varepsilon, \mathbb{T}_\varepsilon \times D_{(L+1)/\varepsilon}) = \frac{E_\varepsilon(u_\varepsilon, \mathbb{T} \times D_{L+1})}{\pi|\log\varepsilon|} \to 2\pi\sqrt{1+L^2}. \tag{13}$$

Remark 5. The solution $U(x) = V_0(x_2, x_3)$ with a straight line vortex satisfies a stronger stability result. Indeed, from [BMR], [Sha], [San2] and [Mi], we know that V_0 is a local (in space) minimizer of the Ginzburg-Landau energy E on $\mathbb{R}^2$, and the only local (in space) minimizers are only, up to a translation and multiplication by a complex of modulus 1, V_0 and $\bar{V_0}$. Therefore, U is also a local (in space) minimizer of E *without the constraint on the momentum.*

Remark 6. Note that we first fix a (large) period $2\pi/\varepsilon$ for U_ε, and then the result is that "local" (in space) minimizers of the energy with the constraint on the momentum have vorticity on an helix with the same period. Therefore, the solution U_ε is *not* locally (in space) minimizing for perturbations on $m \in \mathbb{N}$ periods, $m \geq 2$ (with the appropriate constraint on the momentun which is for m periods $P = 2m(\pi\frac{L}{\varepsilon})^2$), since the vorticity of this other minimization problem is an helix of period $2\pi m/\varepsilon$ (and not $2\pi/\varepsilon$). This last minimizer is the one obtained by changing ε for ε/m and L for $L/\sqrt{m}$.

Remark 7. Note finally that there exist maps of finite energy, if we drop the condition of degree one at infinity. Replacing Y_ε by the space X_ε of maps in $L^\infty \cap H^1_{loc}(\mathbb{T}_\varepsilon \times \mathbb{R}^2, \mathbb{C})$ of modulus greater than $1/2$ in a neighborhood of infinity, in which we can define the momentum, the problem of minimizing E in X_ε with the constraint $P = 2(\pi\frac{L}{\varepsilon})^2$ has solutions and the corresponding minimizers of E are, as in Theorem 1 in [BOS], vortex rings ($\frac{2\pi}{\varepsilon}$-periodic in the x_1 variable) of radius L.

As in [BOS], we will not tackle the problem of the stability of the solution U_ε for the Cauchy problem associated to (1). The situation is here even more involved since the solution is of infinite energy. The adapted context should be on a bounded domain (see subsection 1.4). Finally, we would like to mention that vortex helices have been observed numerically in [ABK] for the general Ginzburg-Landau type equation, with real constants c and b,

$$\partial_t A = A - (1+ic)|A|^2A + (1+ib)\Delta A.$$

For $c, b \to +\infty$ (and a suitable renormalization), we recover the standard nonlinear Schrödinger equation (1). In section V of [ABK], in the case $b = 1/\varepsilon \gg 1$, a stable travelling vortex helix is numerically obtained. The boundary conditions are of periodic type as well as the homogeneous Neumann condition. Even for small values of c and b (see [RCK] and [GGNO] for phase twisted initial data), there is convergence to an helical vortex.

1.3 Discussing symmetries

In 3-dimensional space, the existence of travelling vortex rings is proved in [BOS]. This vortex is the circle $\{0\} \times \partial D_1(0)$ and has the cylindrical symmetry. Therefore, it is natural to consider for this problem cylindrically symmetric solutions, that is solutions of the type

$$U(x) = \hat{U}(x_1, r).$$

For our problem, even though we work on a (periodic) cylinder and the condition at infinity is

$$U(x) \simeq \frac{(x_2, x_3)}{|(x_2, x_3)|} = e^{i\theta},$$

thus cylindrically symmetric, we emphasize that the solution does not have the cylindrical symmetry (except for $L = 0$). The more appropriate symmetry is "helicoidal symmetry", that is

$$U(x) = e^{-i\varepsilon x_1} \hat{U}(\varepsilon x_1 - \theta, r) \qquad \text{or} \qquad U(x) = e^{-i\varepsilon x_1} \hat{U}(\varepsilon x_1 + \theta, r), \tag{14}$$

for a $\hat{U} : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{C}$. A straightforward computation shows that $U(x) = e^{-i\varepsilon x_1} \hat{U}(\varepsilon x_1 - \theta, r)$ (resp. $U(x) = e^{-i\varepsilon x_1} \hat{U}(\varepsilon x_1 + \theta, r)$) precisely means that the vector field $U \times \nabla U$ satisfies the property that its components in the cylindrical basis $(\vec{e}_1, \vec{e}_r, \vec{e}_\theta)$ are constant on each helix $(\alpha, 0, 0) + \mathcal{H}_R$ (resp. $(\alpha, 0, 0) + \tilde{\mathcal{H}}_R$), $R \geq 0$ and $\alpha \in \mathbb{T}$. We may impose the first symmetry, for instance, for the solutions without changing the main ideas of the proofs.

Theorem 3. *For every $L > 0$, there exists $\varepsilon_0(L) > 0$ such that, for every $0 < \varepsilon < \varepsilon_0(L)$, there exists a solution $\mathcal{U}_\varepsilon$ to (2), $\frac{2\pi}{\varepsilon}$-periodic in the x_1 variable, with $C = C(\varepsilon)$ verifying (4), (5) and (6) as $\varepsilon \to 0$, and such that $\mathcal{U}_\varepsilon$ is helicoidally symmetric in the sense that*

$$\mathcal{U}_\varepsilon(x) = e^{-i\varepsilon x_1} \hat{\mathcal{U}}_\varepsilon(\varepsilon x_1 - \theta, r).$$

For the proof, it suffices to work on the subspace of maps with the helicoidal symmetry (14). One may also work with the variables $\varepsilon x_1 - \theta$ and r, changing (2) for a 2-dimensional equation with a degenerate elliptic operator.

Remark 8. We have not been able to prove that the solutions provided by Theorem 1 are helicoidally symmetric. The solutions U_ε and $\mathcal{U}_\varepsilon$ are presumably the same up to a rotation, translation and multiplication by a complex number of modulus one. For the helicoidally symmetric solution $\mathcal{U}_\varepsilon$, we can prove a stability result analogous to Theorem 2. We have however to restrict ourselves to perturbations which are also helicoidally symmetric.

Remark 9. One may obtain the other vortex helix $\tilde{U}_L^*$ imposing the other helicoidal symmetry

$$U(x) = e^{-i\varepsilon x_1} \hat{U}(\varepsilon x_1 + \theta, r).$$

1.4 Link with Euler equation

Let us perform on ψ the Madelung transform

$$\psi = \sqrt{\rho} \exp(i\varphi),$$

which has clear meaning if $|\psi| \neq 0$. We may then rewrite equation (1) in the variables $(\rho, \vec{v} := 2\nabla\varphi)$:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0, \\ \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + \nabla(2\rho) = -\nabla\Big(\frac{|\nabla\rho|^2}{2\rho^2} - \frac{\Delta\rho}{\rho}\Big). \end{cases}$$

Neglecting the last term in the right-hand side, often called "quantum pressure", this system reduces to the Euler equations for compressible ideal fluids, with speed $\vec{v}$ and pressure ρ^2. Concerning the existence of vortex helices solutions for the incompressible Euler equation, that is $\operatorname{div} \vec{v} = 0$ and

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p, \tag{15}$$

we may mention the work of [Du], where the question of global in time solutions with helicoidal symmetry to the incompressible Euler equation is investigated. In this context, helicoidal symmetry means vector fields $\vec{v}$ such that the components in the cylindrical basis ($\vec{e}_1$,$\vec{e}_r$,$\vec{e}_\theta$) are constant on each helix $(\alpha, 0, 0) + \mathcal{H}_R$, $R \geq 0$, $\alpha \in \mathbb{T}$ (this is the condition we impose in Theorem 3) and such that the vector field $\vec{v}$ is orthogonal to every helices $(\alpha, 0, 0) + \mathcal{H}_R$ for $R \geq 0$ and $\alpha \in \mathbb{T}$ (that is $\vec{v} \cdot (\vec{e}_1 + r\vec{e}_\theta) = 0$ for every $x \simeq (x_1, r, \theta)$). We can *not* impose this last condition to $\mathcal{U}_\varepsilon$ in Theorem 3 since $U_L^* \times \nabla U_L^*$ does *not* satisfy it (we will see that $U_L^* \times \nabla U_L^* \simeq \frac{\vec{e}_\theta}{r}$ as $r \to +\infty$). Notice also that we did not require the first condition for the solution U_ε, though the limiting vector field $U_L^* \times \nabla U_L^*$ satisfies this constraint. Note that in dimension 3, the solutions to Euler equation may become singular in finite time. The main point is that imposing the helicoidal symmetry reduces the problem to a two-dimensional problem, for which global in time existence results for the incompressible Euler equation are known. The result of [Du] implies for instance that, given $R > 0$ and an initial vector field $\vec{v}_0$ which has helicoidal symmetry, is divergence free and tangent to the boundary of the cylinder $\mathbb{R} \times D_R$, then there exists a unique solution $\vec{v}$, global in time and divergence free, to (15) with initial datum $\vec{v}_0$. Moreover, $\vec{v}(t, .)$ has helicoidal symmetry for all $t \geq 0$. However, it is stated neither that we may choose $\vec{v}_0$ such that the vorticity concentrates on an helix, nor that the solution may be a travelling wave, and *a fortiori* its propagation speed is not computed. In any cases, the vector field $\vec{v} = U_L^* \times \nabla U_L^*$ is *not* orthogonal to the helices $(\alpha, 0, 0) + \mathcal{H}_R$, $\alpha \in \mathbb{T}$, $R \geq 0$, that is the second hypothesis in [Du] is not satisfied.

Concerning the dynamics of vortex filaments described by a map $X = X(s, t)$, where t is time and s is arclength, the equation governing the motion of X in a perfect inviscid fluid, known as LIE (or LIA) (Localized Induction Equation (or Approximation)), has been derived first by L.S. da Rios (see [dR] and also [R]), and then rediscovered by F.R. Hama ([H]), and reads

$$\partial_t X = \partial_s X \times \partial_s^2 X \tag{16}$$

(see also [KM] for the case where the filament may be self-stretched). Assuming the map X smooth, this equation writes, in the Frenet basis $(\vec{\tau}, \vec{\beta}, \vec{\nu})$,

$$\partial_t X = (\vec{H} \cdot \vec{\nu})\vec{\beta}, \tag{17}$$

where $\vec{H} := \frac{d\vec{\tau}}{ds}$ is the (mean) curvature vector, $\vec{\beta}$ being called the binormal. In the case X is a travelling wave with constant speed $c\vec{e}_1$, i.e. $X(s, t) = Y(s) + ct\vec{e}_1$, we clearly recover equation (10). A motion verifying equation (17) is known as a (smooth) *binormal curvature flow*. The paper [GV] studies self-similar solutions to (17) and shows that this equation is ill-posed and can develop singularities. We mention the work of R.L Jerrard [J1], where the convergence in dimension $N \geq 3$ as $\varepsilon \to 0$ to a (weak) binormal curvature flow is proved for the scaled Gross-Pitaevskii equation (1)

$$i|\log \varepsilon|\partial_t u + \Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2) = 0,$$

if the initial datum is in $1 + H^1(\mathbb{R}^N)$ and has a jacobian concentrated on some round $(N - 2)$-dimensional sphere. In the 2-dimensional case, the situation is different and involves the renormalized energy W introduced in [BBH2]. In [LX] (Theorem 1) (see also [CJ]), it is proved that if u_ε solves

$$i\partial_t u_\varepsilon + \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon(1 - |u_\varepsilon|^2) = 0, \qquad u_\varepsilon^{t=0} = u_\varepsilon^0,$$

(either in a bounded domain with a boundary condition of degree n, either in $\mathbb{R}^2$ with $n = 0$ and u_ε^0 tends to 1 at infinity sufficiently fast), then the linear momentum $\rho\vec{v}$ converges to a solution to Euler equation, provided the vortices of the initial datum are of degree ± 1. We note the different time scalings for $N = 2$ and $N \geq 3$. Moreover (Theorem 2 in [LX]), if the datum u_ε^0 is almost minimizing, that is $E_\varepsilon(u_\varepsilon^0) = \pi n|\log\varepsilon| + \pi W(a(0)) + o(1)$ as $\varepsilon \to 0$, the vortices $a_j(t)$, $1 \leq j \leq n$, obey the so called Kirchhoff law for fluid point vortices

$$\frac{da_j}{dt}(t) = \Big(\frac{\partial W}{\partial a_j^2}, -\frac{\partial W}{\partial a_j^1}\Big), \qquad \text{i.e.} \quad i\frac{da_j}{dt}(t) = 2\frac{\partial W}{\partial a_j},$$

where W is seen as a function of the n complex variables $(a_1, \dots, a_n)$.

From (16), R. Betchov establishes in [B] the intrinsic equations on the curvature K and torsion T of the curve, namely

$$\begin{cases} \partial_t K &= -2\partial_s(KT), \\ \partial_t T + 2T\partial_s T &= \frac{1}{2}\Big(\partial_s K + \frac{\partial_s^3 K}{K} + \frac{(\partial_s K)^3}{K^3} - 2\frac{(\partial_s K)(\partial_s^2 K)}{K^2}\Big), \end{cases}$$

and the helicoidal solution (among others) is exhibited as a solution for T and K constant. However, the corresponding linearized equations are shown to be unstable. These equations have then been solved for vortex filaments without change of form in [K] and here again the helix is shown to be unstable. On the other hand, vortex helices (and other vortex filaments) in an axial flow have been experimentally observed by T. Maxworthy, E.J. Hopfinger and L.G. Redekopp in [MHR]. Later, Y. Fukumoto and T. Miyazaki in [FM] have studied the stability of vortex filaments in axial flows, which changes LIE for a more complex equation, and have shown that vortex helices are stable under some conditions on the axial flow and the torsion of the helix, corroborating the observations of [MHR]. It would be interesting, by analogy with the the study of [FM], to investigate the stability of the solution U_ε in the context of rotating superfluids. In this case, the energy in the rotating frame, taking into account the Coriolis force, denoting $\vec{\omega} = \omega\vec{e}_1$ the rotation vector and ω the angular velocity around the axis x_1, is

$$\frac{1}{2}\int_\Omega |\nabla u + iu\vec{\omega} \times x|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2},$$

where $\Omega \subset \mathbb{R}^3$ is, for instance, a cylinder of axis x_1. In this context, a homogeneous Neumann condition should be prescribed, at least on the lateral boundary of the cylinder.

For superfluids like Helium II, an equation analoguous to LIA can be derived (see for instance [Do] or [Sam]). This equation, called the Schwarz's equation, writes

$$\partial_t X = \vec{v}_i + \vec{v}_{ap} + \alpha\partial_s X \times (\vec{v}_n - \vec{v}_{ap} - \vec{v}_i) - \alpha'\partial_s X \times \Big(\partial_s X \times (\vec{v}_n - \vec{v}_{ap} - \vec{v}_i)\Big),$$

where $\vec{v}_{ap}$ is the applied flow, $\vec{v}_n$ is the velocity of the normal fluid, and $\vec{v}_i$ is the velocity induced by the filament. This term is given by the Biot-Savart law, and is approached in the Localized Induction Approximation by (up to a physical constant factor) $\partial_s X \times \partial_s^2 X$. The constants α and α' are friction coefficients between the normal fluid and the superfluid. If there is no friction, $\alpha = \alpha' = 0$. The LIA is not always a good approximation, especially when parts of the filaments get close one another or self-intersect. However, for our helicoidal vortex, (16) remains a good approximation.

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2 Strategy of the proof

In this Section, we give the scheme of the proof of Theorems 1 and 2. The idea is to approximate the problem on cylinders of growing diameter. We will point out the problems related to the fact that the solution is of infinite energy. We will give the precise proofs in Sections 4 to 8. In the sequel, C_L will denote a constant depending on L only.

2.1 Setting

We will denote $(\vec{e}_1, \vec{e}_r, \vec{e}_\theta)$ the usual cylindrical basis. The next Lemma states that $\vec{v}$ behaves like $\frac{\vec{e}_\theta}{r} =$ "$\nabla\theta$" at infinity, that is U_L^* is expected to be close to $e^{i\theta}$ as $r \to +\infty$.

Lemma 1. *We have*

$$\int_{\mathbb{T}\times\{r\geq L+1\}} \|\vec{v} - \frac{\vec{e}_\theta}{r}\|^2 < +\infty.$$

Since both energy and momentum are not easily defined, we approximate the problem on cylinders

$$\Omega_n := \mathbb{T} \times D_n,$$

for $n \in \mathbb{N}^*$. In view of Lemma 1 above, we expect that the behavior of the solution U_ε is

$$U_\varepsilon(x) \simeq \tfrac{(x_2,x_3)}{|(x_2,x_3)|} \qquad \text{as } |(x_2, x_3)| \to +\infty,$$

thus we naturally impose a boundary condition on $\partial\Omega_n = \mathbb{T} \times \partial D_n$

$$g : x \mapsto e^{i\theta} = \frac{(x_2, x_3)}{|(x_2, x_3)|}.$$

The function g is well-defined and smooth on $\mathbb{T} \times (\mathbb{R}^2 \setminus \{0\})$, and will impose a degree one on the boundary (in the (x_2, x_3)-plane). We work on the affine space

$$X_n := H_g^1(\Omega_n, \mathbb{C})$$

(note that $H^1(\Omega_n)$ embeds compactly in $L^4(\Omega_n)$). Since Ω_n is bounded, we can define the momentum

$$p(u) := \int_{\Omega_n} (iu, \partial_1 u)\ dx.$$

We will work in the (clearly non void) set

$$\Gamma_n := \{u \in X_n,\ p(u) = 2\pi^2 L^2\},$$

where the momentum $2\pi^2 L^2$ has to be understood as $2\pi \times (\pi L^2)$, that is 2π times the area of a disk of radius L. We then consider the minimization problem

$$(\boldsymbol{\mathcal{P}_\varepsilon^n}) \qquad \boldsymbol{I_\varepsilon^n := \inf_{u\in\Gamma_n} E_\varepsilon(u).}$$

The existence of a minimizer for the problem $(\mathcal{P}_\varepsilon^n)$ is straightforward.

Proposition 2. *There exist a minimizer $u_{\varepsilon,n} \in \Gamma_n$ for the problem $(\mathcal{P}_\varepsilon^n)$ and a constant $c_{\varepsilon,n} \in \mathbb{R}$ such that $u_{\varepsilon,n}$ satisfies* (9)*, that is*

$$ic_{\varepsilon,n}|\log\varepsilon|\partial_1 u_{\varepsilon,n} = \Delta u_{\varepsilon,n} + \frac{1}{\varepsilon^2} u_{\varepsilon,n}(1 - |u_{\varepsilon,n}|^2) \qquad \text{in } \Omega_n. \tag{18}$$

Moreover, the following upper bounds hold for $n \geq L + 2$

$$I_\varepsilon^n = E_\varepsilon(u_{\varepsilon,n}) \leq 2\pi^2 \log n + 2\pi^2\sqrt{1 + L^2}|\log\varepsilon| + C_L \tag{19}$$

and

$$\frac{1}{2}\int_{\mathbb{T}\times D_n} |\partial_1 u_{\varepsilon,n}|^2 + \big|\nabla_{2,3}|u_{\varepsilon,n}|\big|^2 + \frac{1}{2\varepsilon^2}(1 - |u_{\varepsilon,n}|^2)^2\ dx \leq C_L|\log\varepsilon|. \tag{20}$$

Remark 2.1. The constant $\frac{c_{\varepsilon,n}}{2}|\log\varepsilon| \in \mathbb{R}$ in (18) is the Lagrange multiplier, due to the constraint on the momentum. In (19), the term $2\pi^2\log n$ is the diverging term due to the degree one at infinity. The term $2\pi^2\sqrt{1+L^2}|\log\varepsilon|$ is the one that will bound the length of the singular set. The upper bound (20) is deduced from a lower bound taking into account this degree at infinity.

An important remark has to be made concerning the momentum. We recall the definition of the jacobian for $u \in H^1(\Omega,\mathbb{C})$

$$Ju := \frac{1}{2}\, d(u\times du) = \sum_{i<j}(\partial_i u \times \partial_j u)\ dx_i \wedge dx_j,$$

and define ξ as the 2-form

$$\xi := x_2\ dx_1 \wedge dx_2 + x_3\ dx_1 \wedge dx_3 = r dx_1 \wedge dr,$$

which appears when we integrate by parts the momentum, since $d^*\xi = 2dx_1$. Indeed, in view of the boundary condition, we have $(u_{\varepsilon,n}\times du_{\varepsilon,n})_\top = (g\times dg)_\top = d\theta = \frac{1}{n^2}\star\xi$ on $\mathbb{T}\times\partial D_n$, thus integration by parts yields

$$p(u_{\varepsilon,n}) = \int_{\Omega_n}\langle Ju_{\varepsilon,n},\xi\rangle.$$

Our first aim is to bound the speed $c_{\varepsilon,n}$ to be able to use the equation. However, this bound does not rely directly, as in [BOS], on Pohozaev identity, since the left hand side of (9) is of the order of $\log n$. A strategy could be to try to localize the energy, or the momentum, but this can not be done with the use of the equation, which requires precisely a bound on $c_{\varepsilon,n}$. To break this vicious circle, our approach will be to use a regularization technique.

2.2 The regularized problem

We consider the following parabolic regularization problem. First, define

$$\tilde u(x) := \begin{cases} u_{\varepsilon,n}(x) & \text{if } |u_{\varepsilon,n}(x)|\le 1,\\ \frac{u_{\varepsilon,n}(x)}{|u_{\varepsilon,n}(x)|} & \text{if not.}\end{cases}$$

It is clear that

$$|\tilde u|_\infty \le 1 \qquad \text{and} \qquad E_\varepsilon(\tilde u) \le E_\varepsilon(u_{\varepsilon,n}) = I^n_\varepsilon.$$

We then consider the minimization problem

$$(\boldsymbol{\mathcal{R}^n_\varepsilon}) \qquad \inf_{v\in X_n} E_\varepsilon(v) + \int_{\Omega_n}\frac{|\tilde u - v|^2}{2\varepsilon},$$

for which the existence of a minimizer $v_{\varepsilon,n}$ is straightforward. Its first properties are given in the following lemma.

Lemma 2.1. *The map $v_{\varepsilon,n}$ satisfies for $n \ge L+2$ and $0<\varepsilon<1/4$*

$$E_\varepsilon(v_{\varepsilon,n}) + \int_{\Omega_n}\frac{|\tilde u - v_{\varepsilon,n}|^2}{2\varepsilon} \le I^n_\varepsilon \le 2\pi^2\log n + 2\pi^2\sqrt{1+L^2}|\log\varepsilon| + C_L. \tag{21}$$

It satisfies also the equation

$$\Delta v_{\varepsilon,n} + \frac{1}{\varepsilon^2}v_{\varepsilon,n}(1-|v_{\varepsilon,n}|^2) = \frac{1}{\varepsilon}(v_{\varepsilon,n}-\tilde u) \qquad \text{on } \Omega_n \tag{22}$$

and for a constant C_0 independent of ε, n and L,

$$|v_{\varepsilon,n}|_\infty \le 1, \qquad |\nabla v_{\varepsilon,n}|_\infty \le \frac{C_0}{\varepsilon}. \tag{23}$$

Proof of Lemma 2.1. To obtain (21), just take $\tilde{u}$ as a comparison map, and (22) is the Euler equation for $(\mathcal{R}_\varepsilon^n)$. Consider the orthogonal projection of $v_{\varepsilon,n}$ on the disk $\bar{D}_1$

$$\tilde{v}(x) := \begin{cases} v_{\varepsilon,n}(x) & \text{if } |v_{\varepsilon,n}(x)| \leq 1, \\ \frac{v_{\varepsilon,n}(x)}{|v_{\varepsilon,n}(x)|} & \text{if not.} \end{cases}$$

By convexity of $\bar{D}_1$, we have $|\nabla\tilde{v}| \leq |\nabla v_{\varepsilon,n}|$, $|\tilde{v}-\tilde{u}| \leq |v_{\varepsilon,n}-\tilde{u}|$ and

$$\int_{\Omega_n} (1-|\tilde{v}|^2)^2 = \int_{\{|v_{\varepsilon,n}|\leq 1\}} (1-|v_{\varepsilon,n}|^2)^2 \leq \int_{\Omega_n} (1-|v_{\varepsilon,n}|^2)^2,$$

with strict inequality unless $|v_{\varepsilon,n}| \leq 1$ a.e. in Ω_n. Therefore,

$$E_\varepsilon(\tilde{v}) + \int_{\Omega_n} \frac{|\tilde{u}-\tilde{v}|^2}{2\varepsilon} \leq E_\varepsilon(v_{\varepsilon,n}) + \int_{\Omega_n} \frac{|\tilde{u}-v_{\varepsilon,n}|^2}{2\varepsilon}$$

with strict inequality unless $|v_{\varepsilon,n}|_\infty \leq 1$. Since $v_{\varepsilon,n}$ is minimizing and $\tilde{v} = g$ on $\partial\Omega_n$, $|v_{\varepsilon,n}|_\infty \leq 1$. For the estimate on the gradient, consider the scaled map $\hat{v}(x) := v_{\varepsilon,n}(\varepsilon x)$, which satisfies

$$\Delta\hat{v} + \hat{v}(1-|\hat{v}|^2) = \varepsilon(\hat{v} - \tilde{u}(\varepsilon x)),$$

and the estimate on the gradient comes from classical estimates for elliptic equations since $\hat{v} = g = e^{i\theta}$ on $\mathbb{T}\times\partial D_{n/\varepsilon}$, $|\hat{v}|_\infty \leq 1$ and $|\tilde{u}(\varepsilon.)|_\infty \leq 1$. $\square$

The advantage of working with $v_{\varepsilon,n}$ instead of $u_{\varepsilon,n}$ is that it is close enough to $u_{\varepsilon,n}$ and satisfies the bound (23) for the gradient, whereas $u_{\varepsilon,n}$ does not since $c_{\varepsilon,n}$ is not bounded yet. The next lemma states that the two expressions integrated for the calculus of the momentum of $v_{\varepsilon,n}$ and $u_{\varepsilon,n}$ are close (say roughly in $L^1(\Omega_n)$), hence it suffices to localize the first one to localize the second one.

Lemma 2. *For a constant C_L independent of $0<\varepsilon<1/4$ and $n \geq L+2$ and for every (measurable) set $B \subset D_n$, we have*

$$\Big|\int_{\mathbb{T}\times B} (iv_{\varepsilon,n}, \partial_1 v_{\varepsilon,n}) - \int_{\mathbb{T}\times B} (iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n})\Big| \leq C_L\sqrt{\varepsilon}|\log\varepsilon|. \tag{24}$$

One of the main advantage of working with $v_{\varepsilon,n}$ is that we will be able to localize sufficiently the singular set of $v_{\varepsilon,n}$ in order to bound the speed $c_{\varepsilon,n}$. This will use a result of [San2] (see also [J2]) concerning lower bounds for the Ginzburg-Landau energy. We denote

$$C(a,R) := \mathbb{T}\times D_R(a) \qquad \text{and} \qquad \check{C}(a,R) := \Omega_n \cap C(a,R),$$

and prove that the singular set $\{|v_{\varepsilon,n}| \leq 1/2\}$ of $v_{\varepsilon,n}$ is included in some not too large cylinders.

Lemma 3. *For $0<\varepsilon<1/4$ and $n \geq C_L|\log\varepsilon|$, there exists a finite family of cylinders $(C(a_j,r_j))_{j\in J}$, depending on ε and n, such that*

$$\{|v_{\varepsilon,n}| \leq 1/2\} \subset \cup_{j\in J}\check{C}(a_j,r_j) \qquad \textit{and} \qquad \sum_{j\in J} r_j \leq C_L|\log\varepsilon|.$$

Since now the singular set of $v_{\varepsilon,n}$ is localized, thus the momentum of $u_{\varepsilon,n}$, we are now in position to control the Lagrange multiplier, using carefuly a Pohozaev-type identity.

Proposition 3. *We have, for a constant $K(L)$ depending on L but independent of $0<\varepsilon<\varepsilon_0(L)$ and $n \geq C_L|\log\varepsilon|^2$,*

$$|c_{\varepsilon,n}| \leq K(L). \tag{25}$$

2.3 First estimates

Since now, $c_{\varepsilon,n}$ is bounded, we can make use of equation (9) and derive the first estimates for $u_{\varepsilon,n}$.

Lemma 4. *The function $u_{\varepsilon,n}$ satisfies the L^∞ bounds*

$$|u_{\varepsilon,n}|^2_{L^\infty(\Omega_n)} \leq 1 + (\frac{c_{\varepsilon,n}}{2}\varepsilon|\log\varepsilon|)^2 \leq C_L \qquad \text{and} \qquad |\nabla u_{\varepsilon,n}|_{L^\infty(\Omega_n)} \leq \frac{C_L}{\varepsilon}. \tag{26}$$

Proof of Lemma 4. We argue as in Lemma 3 in [BOS]. Note $u_{\varepsilon,n} = u$ for simplicity. From (9), we deduce

$$\begin{aligned}
\Delta|u|^2 &= 2(u,\Delta u) + 2|\nabla u|^2 = -2\varepsilon^{-2}|u|^2(1-|u|^2) + 2c_{\varepsilon,n}|\log\varepsilon|(u, i\partial_1 u) + 2|\nabla u|^2 \\
&\geq -\frac{2}{\varepsilon^2}|u|^2(1-|u|^2) - 2|c_{\varepsilon,n}|\cdot|\log\varepsilon|\cdot|u|\cdot|\nabla u| + 2|\nabla u|^2 \\
&= -\frac{2}{\varepsilon^2}|u|^2(1-|u|^2) + \Big(\sqrt{2}|\nabla u| - \frac{|c_{\varepsilon,n}|}{\sqrt{2}}|u|\cdot|\log\varepsilon|\Big)^2 - \frac{|c_{\varepsilon,n}|^2}{2}|u|^2\cdot|\log\varepsilon|^2 \\
&\geq -\frac{2}{\varepsilon^2}|u|^2\Big(1-|u|^2 + (\frac{c_{\varepsilon,n}}{2}\varepsilon|\log\varepsilon|)^2\Big).
\end{aligned}$$

Since $|u| = 1$ on $\mathbb{T}\times\partial D_n$, the function $w = 1 - |u|^2 + (\frac{c_{\varepsilon,n}}{2}\varepsilon|\log\varepsilon|)^2$ satisfies

$$\begin{aligned}
-\Delta w + \frac{2}{\varepsilon^2}|u|^2 w \geq 0 &\qquad \text{in } \Omega_n, \\
w \geq 0 &\qquad \text{on } \partial\Omega_n,
\end{aligned}$$

and by the maximum principle, we deduce $w \geq 0$ in Ω_n, which is the first estimate. Concerning the second one, we consider the scaled map $\hat{u}(x) := u(\varepsilon x)$, which satisfies

$$\begin{aligned}
\Delta\hat{u} + \hat{u}(1-|\hat{u}|^2) = ic_{\varepsilon,n}\varepsilon|\log\varepsilon|\partial_1\hat{u} &\qquad \text{in } \mathbb{T}\times D_{n/\varepsilon}, \\
\hat{u} = e^{i\theta} &\qquad \text{on } \mathbb{T}\times\partial D_{n/\varepsilon}.
\end{aligned}$$

By standard elliptic estimates (see [GT]),

$$|\nabla\hat{u}|_{L^\infty} \leq C_L$$

and we conclude by scaling back. □

We will also use the following Clearing-Out (or η-ellipticity) result.

Theorem 4. *Let $\sigma > 0$ be given. Then, there exist $\eta > 0$ and $\varepsilon_0 > 0$, depending only on σ and L, such that, for $x_0 \in \bar{\Omega}_n$, $0 < \varepsilon \leq \varepsilon_0$ and $\varepsilon^\mu \leq r \leq 1$ (with $n \geq C_L|\log\varepsilon|^2$ and $\mu \in (0,1)$ absolute), if*

$$r^{-1}E_\varepsilon(u_{\varepsilon,n}, B_r(x_0)) \leq \eta|\log\varepsilon|,$$

then

$$|u_{\varepsilon,n}(x_0)| \geq 1 - \sigma.$$

This result is an easy consequence of Theorem 2 in [BOS] inside the domain Ω_n and Theorem 2 in [C2] near the boundary, since the boundary condition $g(x) = e^{i\theta}$ is uniformly smooth for $n \geq 1$ and of modulus 1, and the constants in [C2] involving the curvature of the boundary $\partial\Omega_n = \mathbb{T}\times\partial D_n$ are uniformly bounded in n.

We infer from Theorem 4 the finer localization of the singular set of $u_{\varepsilon,n}$ defined by

$$S^n_\varepsilon := \{|u_{\varepsilon,n}| \leq 1/2\}.$$

Corollary 1. *Let $0 < \varepsilon < \varepsilon_0(L)$ and $n \geq C_L|\log\varepsilon|^2$. There exist $R_0 > 0$ and $l \in \mathbb{N}^*$, depending on L only, and q cylinders $C(a_i, R_0)$ $(1 \leq i \leq q)$, with $q \leq l$, such that the cylinders $C(a_i, 8R_0)$ $(1 \leq i \leq q)$ are mutually disjoint,*

$$S_\varepsilon^n \subset \cup_{i=1}^q C(a_i, R_0),$$

$$\int_{\cup_{i=1}^q \check{C}(a_i,R_0)} e_\varepsilon(u_{\varepsilon,n}) \leq 2\pi^2\sqrt{1+L^2}|\log\varepsilon| + C_L. \tag{27}$$

Moreover, for every $a \in \mathbb{R}^2$, we have

$$\int_{\check{C}(a,8R_0)} e_\varepsilon(u_{\varepsilon,n}) \leq C_L|\log\varepsilon|. \tag{28}$$

We then define a rectifiable 1-dimensional integral current by the mean of the Γ-convergence results of [JS] and [ABO]. This is possible thanks to the localization given in Corollary 1. Nevertheless, we will be compelled to work with

$$\tilde{u}_{\varepsilon,n}(x) := \begin{cases} 2u_{\varepsilon,n}(x) & \text{if } |u_{\varepsilon,n}(x)| \leq \frac{1}{2}, \\ \frac{u_{\varepsilon,n}(x)}{|u_{\varepsilon,n}(x)|} & \text{if not.} \end{cases}$$

It is clear that $J\tilde{u}_{\varepsilon,n}$ is supported in $\cup_{i=1}^q C(a_i, R_0)$, since outside this set, $\tilde{u}_{\varepsilon,n}$ is smooth and of modulus 1, thus two partial derivatives of $\tilde{u}_{\varepsilon,n}$ are both tangent to $\mathbb{S}^1$ at $\tilde{u}_{\varepsilon,n}$, hence are colinear. Notice that, comparing with Lemma 3.1 in [BOS], we do not know yet that $J\tilde{u}_{\varepsilon,n}$ and $Ju_{\varepsilon,n}$ are close globally in Ω_n, since the energy diverges as $n \to +\infty$. We define (2π times) the flux of $\vec{e}_1$ through a 1-dimensional current T with compact support, by

$$\mathcal{F}(T) := \pi\langle T, \star\xi\rangle.$$

The name flux becomes clear when $T = \partial R$ is a boundary, since then integration by parts yields

$$\mathcal{F}(T) = \pi\langle \partial R, \star\xi\rangle = \pi\langle R, \star d^*\xi\rangle = 2\pi\langle R, \star dx_1\rangle.$$

It has been noticed in [BS] and [BOS] that the momentum p may be interpreted as the flux $\mathcal{F}(T)$. In our context, since T is x_1-periodic, the projection R of T on the (x_2, x_3)-plane is a closed "curve" R and $\mathcal{F}(T)$ is interpreted as the flux of $\vec{e}_1$ through the surface enclosed by R.

Lemma 5. *For $n \geq C_L|\log\varepsilon|^2$ and $0 < \varepsilon < \varepsilon_0(L)$, there exists a 1-dimensional integral current $T_{\varepsilon,n}$ without boundary, supported in the cylinders $C(a_i, R_0)$, $1 \leq i \leq q$, such that*

i) $\|J\tilde{u}_{\varepsilon,n} - \pi T_{\varepsilon,n}\|_{[\mathcal{C}_c^{0,1}(\Omega_n)]^*} \leq r(\varepsilon)$,

ii) $|p(u_{\varepsilon,n}) - \mathcal{F}(T_{\varepsilon,n})| \leq r(\varepsilon)$,

iii) $\mathbf{M}(T_{\varepsilon,n}) \leq \frac{E_\varepsilon(u_{\varepsilon,n}, \cup_{i=1}^q C(a_i,8R_0))}{\pi|\log\varepsilon|} + r(\varepsilon)$,

where $r(\varepsilon)$ is a function which tends to 0 if $\varepsilon \to 0$ uniformly for $n \geq C_L|\log\varepsilon|^2$. Moreover, for every $a \in \mathbb{R}^2$,

$$||J\tilde{u}_{\varepsilon,n} - Ju_{\varepsilon,n}||_{[\mathcal{C}_c^{0,1}(\check{C}(a,8R_0))]^*} \leq C_L\varepsilon|\log\varepsilon|. \tag{29}$$

2.4 An isoperimetric problem

As in [BOS], we characterize the limiting singular set with the help of the equality case in an isoperimetric type inequality. We define the projection on the x_1-axis of T

$$Pr_1(T) := \langle T, dx_2 \wedge dx_3 \rangle \in \mathbb{R}.$$

For their purpose, in [BOS], it is made use of the standard isoperimetric inequality. We will make use of the isoperimetric type problem given in the next lemma.

Lemma 6. *Let $L \geq 0$ and T be a 1-dimensional integral current in $\mathbb{T} \times \mathbb{R}^2$ compactly supported and without boundary such that*

$$Pr_1(T) = 2\pi \quad \text{and} \quad \mathcal{F}(T) = 2\pi^2 L^2.$$

Then,

$$\mathbf{M}(T) \geq 2\pi\sqrt{1+L^2}.$$

If, moreover, we assume $\mathbf{M}(T) \leq 2\pi\sqrt{1+L^2}$, then there exists a translation τ in $\mathbb{T} \times \mathbb{R}^2$ such that

$$\tau(T) = \mathcal{H}_L.$$

Remark 2.2. We emphasize that this is the exact values of the flux $\mathcal{F}(T)$ and $Pr_1(T)$ that fix the orientations and thus the exact helix. Indeed, one could have thought about the helix $\mathcal{H}_L$ with reverse orientation, but this would have changed the sign of $Pr_1(T)$ or choose the helix $\tilde{\mathcal{H}}_L$ (with an orientation to be chosen) already mentionned, but this time, $\mathcal{F}(T)$ would have changed sign since $Pr_1(T) > 0$ imposes the orientation of $\tilde{\mathcal{H}}_L$ to be the one of the parametrization $\mathbb{T} \ni \theta \mapsto (\theta, L\cos\theta, -L\sin\theta))$.

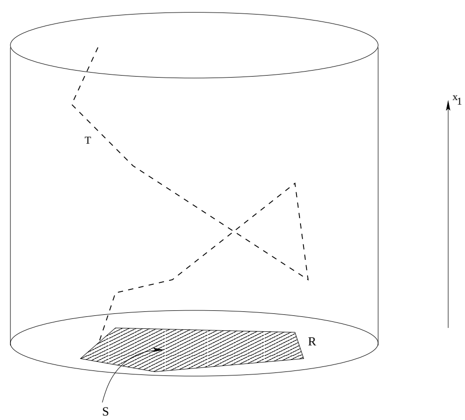

Figure 2: **The isoperimetric problem**

Remark 2.3. Let us explain this isoperimetric type problem. The "curve" T is periodic in the x_1 variable, thus the projection R of T on the (x_2, x_3)-plane is a closed curve enclosing an algebraic surface S. The constraint on the flux imposes S to have an area at least πL^2. By the isoperimetric inequality in $\mathbb{R}^2$, R has a length at least $2\pi L$. Moreover, T has a length at least 2π in the x_1 variable, hence T has length at least $2\pi\sqrt{1+L^2}$. The equality case imposes equality in the isoperimetric inequality, and then R is a circle of radius L. We then conclude that T is the helix $\mathcal{H}_L$.

This result combined with Corollary 1 enables us to give a precise location of the singular set of $u_{\varepsilon,n}$, included in a single cylinder, which concentrates the $|\log\varepsilon|$ term of the energy. The diverging term $\log n$ is non-local and entirely outside the cylinder. We can also give the asymptotics for the energy around and outside the cylinder containing the helix.

Proposition 4. *There exist $\varepsilon_0 > 0$ and $R_0 > 0$ such that, for all $n \geq C_L|\log\varepsilon|^2$ and $0 < \varepsilon < \varepsilon_0$, there exists $b = b(\varepsilon, n) \in D_n$ such that*

$$S_\varepsilon^n \subset \mathbb{T} \times D_{R_0}(b) = C_{R_0}(b), \tag{30}$$

$$E_\varepsilon(u_{\varepsilon,n}, \check{C}_{R_0}(b)) = 2\pi^2\sqrt{1+L^2}|\log\varepsilon| + r(\varepsilon)|\log\varepsilon|, \tag{31}$$

$$E_\varepsilon(u_{\varepsilon,n}, \Omega_n \setminus \check{C}_{R_0}(b)) \leq 2\pi^2 \log n + r(\varepsilon)|\log\varepsilon|. \tag{32}$$

Moreover,

$$c_{\varepsilon,n} = \frac{1}{\sqrt{1+L^2}} + r(\varepsilon), \tag{33}$$

and up to a translation in the x_1 variable, denoting τ_{-b} the translation of vector $(0, -b) \in \mathbb{T} \times \mathbb{R}^2$,

$$||\tau_{-b}T_{\varepsilon,n} - \vec{\mathcal{H}}_L||_{[\mathcal{C}_c^{0,1}]^*} \leq r(\varepsilon). \tag{34}$$

Statement (34) in Proposition 4 will imply that $u_{\varepsilon,n}$ is close to the solution U_L^* we want. In the next Section, we complete the proofs of Theorems 1 and 2 and of Proposition 1 letting $n \to +\infty$. In Section 4, we give the proof of Proposition 2. Lemmas 2 and 3 are proved in Section 5, Proposition 3, Corollary 1 and Lemma 5 in Section 6. The proof of Proposition 4 is given in Section 7. Finally, the proofs of the auxiliary Lemmas 1 and 6 are given in Section 8.

3 Proofs of Theorems 1 and 2 completed

3.1 Limits of growing cylinders

Before going further, we prove that b is not too close from the boundary.

Lemma 3.1. *There exists $0 < \gamma < 1$ such that, for $n \geq e^{1/\varepsilon}$ and $0 < \varepsilon < \varepsilon_0(L)$ sufficiently small,*

$$\|b(\varepsilon, n)\| \leq \gamma n.$$

Remark 3.1. Though it might be, we do not prove that the helix $T_{\varepsilon,n}$ is centered around the x_1-axis, that is $\|b\| = r(\varepsilon)$, or even $\|b\| \leq C_L$. However, in Lemma 3.6 below, we will prove that

$$\lim_{n\to+\infty} \frac{\|b\|}{n} = 0.$$

Proof of Lemma 3.1. From (19) and (20), we deduce by averaging that there exists a $x_1 \in \mathbb{T}$ such that

$$\frac{1}{2}\int_{D_n} |\nabla_{2,3}u_{\varepsilon,n}(x_1, .)|^2 + \frac{(1-|u_{\varepsilon,n}(x_1, .)|^2)^2}{2\varepsilon^2} \leq \pi \log n + C_L|\log\varepsilon|.$$

Consider the scaled map $\hat{u} : D_1 \to \mathbb{C}$ defined by

$$\hat{u}(y) := u_{\varepsilon,n}(x_1, ny).$$

Then $\hat{u} = e^{i\theta}$ on ∂D_1 and, denoting $\delta := \varepsilon/n$, we have by scaling

$$\frac{1}{2}\int_{D_1} |\nabla\hat{u}|^2 + \frac{(1-|\hat{u}|^2)^2}{2\delta^2} \leq \pi|\log\delta| + C_L|\log\varepsilon| \leq \pi|\log\delta|(1 + o(1))$$

since by hypothesis, $n \geq e^{1/\varepsilon}$. Therefore, we may apply the results of [J2] or [San2] stating that $\hat{u}$ has only one "bad disk", the center of which is clearly $\frac{b}{n} + \mathcal{O}(\delta)$. Adapting the arguments of chapter I and Lemma VI.1 in chapter VI in [BBH2], we infer that the vortex can not be on the boundary, for otherwise the energy would be $\geq 2\pi|\log\delta|(1+o(1))$. Therefore, $\|\frac{b}{n}\| \leq \gamma < 1$ for ε small enough and $n \geq e^{1/\varepsilon}$. $\square$

From now on, we translate the problem so that the helix is centered around the x_1 axis, that is we consider $u_{\varepsilon,n} \circ \tau_{-b} : \Omega_n(b) := \mathbb{T} \times D_n(b) \to \mathbb{C}$ instead of $u_{\varepsilon,n}$. In particular, (x_1, r, θ) will now refer to cylindrical coordinates centered around the singular helix. From Lemma 3.1, we have $\|b\| \leq \gamma n$, and therefore $dist(0, \partial\Omega_n(b)) \geq (1-\gamma)n \to +\infty$ as $n \to +\infty$.

We let now $n \to +\infty$ with fixed (small) ε to obtain a solution u_ε on $\mathbb{T} \times \mathbb{R}^2$. To extract a subsequence as $n \to +\infty$, we use the local boundness for $u_{\varepsilon,n}$ in H^1_{loc} given in (28) in Corollary 1. As a consequence, up to a subsequence, we may assume, as $n \to +\infty$,

$$u_{\varepsilon,n} \rightharpoonup u_\varepsilon \quad \text{in } H^1_{loc}(\mathbb{T} \times \mathbb{R}^2, \mathbb{C}) \qquad \text{and} \qquad u_{\varepsilon,n} \to u_\varepsilon \quad \text{in } L^4_{loc}(\mathbb{T} \times \mathbb{R}^2, \mathbb{C}) \text{ and a.e.}$$

and for every $a \in \mathbb{R}^2$

$$E_\varepsilon(u_\varepsilon, C_{R_0}(a)) \leq C_L|\log\varepsilon|. \tag{35}$$

Moreover, by ellipticity of equation (9),

$$u_{\varepsilon,n} \to u_\varepsilon \quad \text{in } H^1_{loc}(\mathbb{T} \times \mathbb{R}^2, \mathbb{C}) \quad \text{as } n \to +\infty.$$

Note also that $(u_{\varepsilon,n})_n$ is bounded in L^∞, so is u_ε, and since $c_{\varepsilon,n}$ is bounded independently of ε and n, we may assume also the existence of the limit

$$c_\varepsilon = \lim_{n \to +\infty} c_{\varepsilon,n} \in \mathbb{R}.$$

We may then pass to the limit in equation (9) to obtain that u_ε satisfies (9) in $\mathbb{T} \times \mathbb{R}^2$ with speed c_ε. Note that, in view of (33) in Proposition 4, the assertion (4) concerning the speed in Theorem 1 is proved.

3.2 Bounds in $W^{1,p}_{loc}(\mathbb{T} \times \mathbb{R}^2)$ and in $\mathcal{C}^k_{loc}$ away from $\mathcal{H}_L$.

The first step is to establish bounds for u_ε in $W^{1,p}_{loc}(\mathbb{T} \times \mathbb{R}^2)$.

Lemma 3.2. *Let $1 \leq p < \frac{3}{2}$. We have, for every $a \in \mathbb{R}^2$,*

$$\int_{\check{C}_{R_0}(a)} |\nabla u_{\varepsilon,n}|^p \leq C_L(p).$$

Proof of Lemma 3.2. The proof of Lemma 3.2 follows exactly the lines of Step 3 of Appendix C in [BOS]. This uses the confinement property of the jacobian $J\tilde{u}$ in the cylinder C_{R_0} and the bound for the energy of the Dirichlet datum (of modulus 1) $\int_{\partial\Omega_n} |\nabla g|^2 = 4\pi^2/n \leq 4\pi^2$. The only difference is that the Hodge-de Rham decomposition (see (C.19) there) of $u_{\varepsilon,n} \times du_{\varepsilon,n}$ on $\Omega_n(b)$ now writes, since $\mathbb{T}$ is not simply connected,

$$u_{\varepsilon,n} \times du_{\varepsilon,n} = d\varphi + d^*\psi + \alpha dx_1, \tag{36}$$

for a constant $\alpha \in \mathbb{R}$. This constant is easily controlled since $\alpha = \frac{L^2}{n^2}$. Indeed, from (36), we infer

$$2\pi^2 n^2 \alpha = |\Omega_n(b)|\alpha = \int_{\Omega_n(b)} \langle u_{\varepsilon,n} \times du_{\varepsilon,n}, dx_1\rangle = \int_{\Omega_n(b)} (iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}) = p(u_{\varepsilon,n}) = 2\pi^2 L^2,$$

and the conclusion follows. $\square$

We then establish uniform bounds in $\mathcal{C}^k_{loc}$ for $u_{\varepsilon,n}$ away from $\mathcal{H}_L$. We follow closely the lines of [BOS] (Steps 6 and 7 in Section 4). These bounds are a direct consequence of the concentration of the density energy (see the proof of Proposition 4) on $\mathcal{H}_L$ as $\varepsilon \to 0$ and $n \geq C_L|\log\varepsilon|^2$ and the bounds $W^{1,p}_{loc}$ just established.

Lemma 3.3. *Let $\bar{B} \subset \mathbb{T}\times\mathbb{R}^2 \setminus \mathcal{H}_L$ be a closed ball and $k \in \mathbb{N}$. Then, for constants $C(k,B,L)$ and $\varepsilon(k,B,L) > 0$ depending only on k, L and a lower bound for the distance from $\bar{B}$ to $\mathcal{H}_L$, we have, for every $0 < \varepsilon < \varepsilon(k,B,L)$, $|u_{\varepsilon,n}| \geq 1/2$ on $\bar{B}$, thus we may write, for a smooth ψ, $u_{\varepsilon,n} = \rho e^{i\psi}$ on $\bar{B}$ and*

i) $\quad \|\nabla\psi\|_{\mathcal{C}^k(\bar{B})} \leq C(k,B,L),$

ii) $\quad \|\frac{2(1-\rho)}{\varepsilon^2} + c_\varepsilon|\log\varepsilon|\partial_1\psi\|_{\mathcal{C}^k(\bar{B})} \leq C(k,B,L).$

In particular,

$$||\nabla\psi||_{L^\infty(\Omega_n(b)\setminus C_{R_0})} \leq C_L \qquad \text{and} \qquad ||1-\rho^2||_{L^\infty(\Omega_n(b)\setminus C_{R_0})} \leq C_L\varepsilon^2|\log\varepsilon|. \tag{37}$$

Proof of Lemma 3.3. We proceed as in Steps 6 and 7 in Section 4 of [BOS]. First, by (32) and Step 4 of the proof of Proposition 4 in Section 7,

$$\Sigma_{\mu_*} = \mathcal{H}_L \qquad \text{and} \qquad E_\varepsilon(u_{\varepsilon,n}, \Omega_n(b)\setminus C_{R_0}) \leq 2\pi^2\log n + r(\varepsilon)|\log\varepsilon|.$$

We apply Lemma 4.4 in subsection 4.3 with $H = D_{R_0}(a)\setminus D_{R_0}$ for an $a \in \mathbb{R}^2\setminus D_{R_0}$ and n sufficiently large (the radius $|H|$ is defined at the beginning of subsection 4.3)

$$\frac{1}{2}\int_{\mathbb{T}\times(D_n(b)\setminus H)} |\nabla_{2,3}u_{\varepsilon,n}|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2} \geq 2\pi^2\log n + 2\pi^2(1-t_*^2)|\log\varepsilon| - 2\pi^2 t_*^2\log(|H|) - C,$$

where C is a constant independent of ε, n and H and

$$t_* := \sqrt{1+\Big(\frac{\pi\varepsilon}{2\sqrt{2}|H|}\Big)^2} - \frac{\pi\varepsilon}{2\sqrt{2}|H|}.$$

Here, $|H| \leq R_0$, and since $t_* \in [0,1]$,

$$E_\varepsilon(u_{\varepsilon,n}, \Omega_n(b)\setminus(C_R(a)\cap C_{R_0})) \geq 2\pi^2\log n - C_L.$$

As a consequence of (32), we infer

$$E_\varepsilon(u_{\varepsilon,n}, C_{R_0}(a)\cap C_{R_0}) \leq r(\varepsilon)|\log\varepsilon|. \tag{38}$$

Thus, for any closed ball $\bar{B} \subset \mathbb{T}\times\mathbb{R}^2\setminus\mathcal{H}_L$, in view of the Clearing-Out result given in Theorem 4 (Theorem 2 of [BOS]), we have for ε sufficiently small (depending on $\bar{B}$) and n sufficiently large,

$$|u_{\varepsilon,n}| \geq \frac{1}{2} \quad \text{in } \bar{B}.$$

Writing in $\bar{B}$ $u_{\varepsilon,n} = \rho e^{i\psi}$, and using Lemma 3.2, we obtain, as in Step 7 in Section 4 of [BOS], for all $k \in \mathbb{N}$, statements *i)* and *ii)* by exactly the same proof. $\square$

We will finally use the following lemma concerning the behavior of the phase at infinity. It is close to statement *ii)* of Theorem 4 in [BOS], but has to be adapted to our problem with a degree one at infinity. We state it only for our solution and not (as in [BOS]) for every solution (on the torus $(\mathbb{R}/(2n\pi\mathbb{Z}))^N$, $N \geq 3$).

Lemma 3.4. *The map $u_{\varepsilon,n}$ writes, for $r \geq R_0$, $u_{\varepsilon,n} = \rho e^{i\varphi + i\theta}$ for a smooth φ x_1-periodic and*

$$\int_{\Omega_n(b)\setminus C_{R_0}} |\nabla\varphi|^2 \leq C_L. \tag{39}$$

Proof of Lemma 3.4. We write, for $r \geq R_0$, $u_{\varepsilon,n} = \rho e^{i\varphi+i\theta}$ and denote

$$v := e^{-i\theta} u_{\varepsilon,n} = \rho e^{i\varphi},$$

which satisfies the equation on $\Omega_n(b) \setminus C_{R_0}$

$$\Delta v - \frac{v}{r^2} + 2i\frac{\partial_\theta v}{r^2} + \frac{1}{\varepsilon^2} v(1-|v|^2) = ic_{\varepsilon,n}|\log\varepsilon|\partial_1 v. \tag{40}$$

We perform a Hodge-de Rham decomposition for $v \times dv$ in $U_n := \Omega_n(b) \setminus C_{R_0}$

$$v \times dv = d\phi + d^*\psi + \alpha dx_1, \tag{41}$$

where ϕ is a smooth function such that $\phi = 0$ in ∂U_n, $\alpha \in \mathbb{R}$ is a constant and ψ is a 2-form such that $d\psi = 0$ and $\psi_\top = 0$ on ∂U_n. Applying the operators d and d^* to (41) and using (40), we deduce the equations in U_n

$$-\Delta\phi = -\frac{c_{\varepsilon,n}}{2}|\log\varepsilon|\partial_1(\rho^2-1) - \frac{\partial_\theta(\rho^2-1)}{r^2}, \tag{42}$$

$$-\Delta\psi = 2Jv. \tag{43}$$

We now turn to estimates for ϕ, ψ and α.

Estimate for $\boldsymbol{\alpha}$. We claim that

$$|\alpha| \leq \frac{r(\varepsilon)}{|U_n|} \leq \frac{r(\varepsilon)}{n^2}. \tag{44}$$

We have, since $v = \rho e^{i\varphi}$ for $r \geq R_0$, with φ periodic in the x_1 variable,

$$|U_n|\alpha = \int_{U_n} \langle v \times dv, dx_1\rangle = \int_{U_n} \rho^2\partial_1\varphi = \int_{U_n} (\rho^2-1)\partial_1\varphi,$$

thus, by Cauchy-Schwarz and using (20) and $\rho \geq 1/2$ for $r \geq R_0$,

$$|U_n| \cdot |\alpha| \leq \frac{\varepsilon}{2}\int_{U_n} \frac{(\rho^2-1)^2}{\varepsilon^2} + |\partial_1\varphi|^2 \leq C_L\varepsilon|\log\varepsilon|$$

and the conclusion follows.

Estimate for $\boldsymbol{\phi}$. We claim that

$$\int_{U_n} |\nabla\phi|^2 \leq C_L\varepsilon^2(1+K(L)|\log\varepsilon|)^2|\log\varepsilon| \leq C_L. \tag{45}$$

Indeed, multiplying (42) by ϕ and integrating yields ($\phi = 0$ on ∂U_n)

$$\begin{aligned}
\int_{U_n} |\nabla\phi|^2 &= -\frac{c_{\varepsilon,n}}{2}|\log\varepsilon|\int_{U_n} \partial_1(\rho^2-1)\phi - \int_{U_n} \frac{\partial_\theta(\rho^2-1)}{r^2}\phi \\
&= \frac{c_{\varepsilon,n}}{2}\varepsilon|\log\varepsilon|\int_{U_n} \frac{\rho^2-1}{\varepsilon}\partial_1\phi - \varepsilon\int_{U_n} \frac{\partial_\theta\phi}{r^2}\frac{\rho^2-1}{\varepsilon} \\
&\leq (K(L)\varepsilon|\log\varepsilon| + \varepsilon)\Big(\int_{U_n} \frac{(\rho^2-1)^2}{\varepsilon^2}\Big)^{1/2}\Big(\int_{U_n} |\nabla\phi|^2\Big)^{1/2} \\
&\leq \varepsilon(K(L)|\log\varepsilon|+1)C_L|\log\varepsilon|^{1/2}\Big(\int_{U_n} |\nabla\phi|^2\Big)^{1/2}
\end{aligned} \tag{46}$$

by (20), which yields the conclusion.

Estimate for ψ. We claim that

$$\int_{U_n} |\nabla\psi|^2 = \int_{U_n} |d^*\psi|^2 \leq C_L + r(\varepsilon)\int_{U_n} |v \times dv|^2. \tag{47}$$

Note that from (41), we have $(d^*\psi)_\top = (v \times dv)_\top - \alpha dx_1$ on ∂U_n. Therefore, ψ is solution of

$$\begin{cases} -\Delta\psi = & 2Jv & \text{in } U_n, \\ \psi_\top = & 0 & \text{on } \partial U_n, \\ (d^*\psi)_\top = & (v\times dv)_\top - \alpha dx_1 & \text{on } \partial U_n. \end{cases}$$

Recalling $|v| = |u_{\varepsilon,n}| \geq 1/2$ in U_n, we define $\tilde{v} := \frac{v}{|v|}$ and consider the solutions ψ_0 and ψ_1 of

$$\begin{cases} -\Delta\psi_0 = & 2J\tilde{v} & \text{in } U_n, \\ (\psi_0)_\top = & 0 & \text{on } \partial U_n, \\ (d^*\psi_0)_\top = & (v\times dv)_\top - \alpha dx_1 & \text{on } \partial U_n \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\psi_1 = & 2(Jv - J\tilde{v}) & \text{in } U_n, \\ (\psi_1)_\top = & 0 & \text{on } \partial U_n, \\ (d^*\psi_1)_\top = & 0 & \text{on } \partial U_n. \end{cases}$$

The existence of ψ_0 and ψ_1 are given by Proposition A.1 in the Appendix of [BBO] and we have $\psi = \psi_0 + \psi_1$. Note also that $J\tilde{v} = 0$ in U_n. Thus, multiplying by ψ_0 and integrating by parts gives

$$\int_{U_n} |\nabla\psi_0|^2 = -\int_{\partial\Omega_n(b)} ((v\times dv)_\top - \alpha dx_1) \wedge (\star\psi_0)_\top - \int_{\partial C_{R_0}} ((v\times dv)_\top - \alpha dx_1) \wedge (\star\psi_0)_\top.$$

Moreover, since the norm of the trace operator from $H^1_\top(D_R)$, with the norm $\|u\|_{H^1_\top} = \|\nabla u\|_{L^2(D_R)}$ (which is equivalent to $\|u\|_{H^1}$), into $L^2(\partial D_R)$ is by scaling $\leq KR^{1/2}$ for an absolute K, we have

$$\int_{U_n} |\nabla\psi_0|^2 \leq K\Big(n\|(v\times dv)_\top - \alpha dx_1\|^2_{L^2(\partial\Omega_n(b))} + R_0\|(v\times dv)_\top - \alpha dx_1\|^2_{L^2(\partial C_{R_0})}\Big). \tag{48}$$

From (44), we have

$$n\|\alpha dx_1\|^2_{L^2(\partial\Omega_n(b))} + R_0\|\alpha dx_1\|^2_{L^2(\partial C_{R_0})} \leq K\alpha^2(n^2 + R_0^2) \leq C_L\frac{r(\varepsilon)}{n^2} = r(\varepsilon),$$

from (37) in Lemma 3.3

$$\|(v\times dv)_\top\|^2_{L^2(\partial C_{R_0})} \leq C_L$$

and finally, in view of the boundary condition, with $z = x_2 + ix_3$,

$$v = e^{-i\theta}\frac{z-b}{|z-b|},$$

we have, since $r \geq (1-\gamma)n$ on $\partial\Omega_n(b)$ by Lemma 3.1

$$\|(v\times dv)_\top\|^2_{L^2(\partial\Omega_n(b))} \leq Kn\Big(\frac{1}{n^2} + \frac{1}{((1-\gamma)n)^2}\Big) \leq \frac{C_L}{n},$$

from which we infer

$$\int_{U_n} |\nabla\psi_0|^2 = \int_{U_n} |d^*\psi_0|^2 \leq r(\varepsilon) + C_L \leq C_L. \tag{49}$$

Moreover, for an absolute (by scaling) constant K

$$\Big(\int_{U_n} |\nabla\psi_1|^2\Big)^{1/2} = \Big(\int_{U_n} |d^*\psi_1|^2\Big)^{1/2} \leq K\sup\Big\{\int_{U_n} \langle Jv - J\tilde{v}, h\rangle, \quad h \in \mathcal{C}^\infty_0(U_n), \quad \int_{U_n} |\nabla h|^2 = 1\Big\}.$$

Since $\tilde{v} \times d\tilde{v} = \rho^{-2} v \times dv$ on U_n, for all $h \in \mathcal{C}_0^\infty(U_n)$ such that $\int_{U_n} |\nabla h|^2 = 1$, we have

$$\Big| \int_{U_n} \langle Jv - J\tilde{v}, h\rangle \Big| = \frac{1}{2} \Big| \int_{U_n} \langle v \times dv - \tilde{v} \times d\tilde{v}, d^* h\rangle \Big|$$
$$\leq \frac{1}{8} \|1-\rho^2\|_{L^\infty(U_n)} \Big(\int_{U_n} |v \times dv|^2 \Big)^{1/2} \Big(\int_{U_n} |\nabla h|^2 \Big)^{1/2} \leq r(\varepsilon) \Big(\int_{U_n} |v \times dv|^2 \Big)^{1/2}$$

in view of (37). As a consequence,

$$\int_{U_n} |\nabla \psi_1|^2 = \int_{U_n} |d^* \psi_1|^2 \leq r(\varepsilon) \int_{U_n} |v \times dv|^2. \tag{50}$$

We deduce (47) from (49) and (50).

Combining (44), (45) and (47) with (41) yields

$$\int_{U_n} |v \times dv|^2 \leq C_L + r(\varepsilon) \int_{U_n} |v \times dv|^2$$

and (39) follows. □

3.3 Convergence of u_ε to U_L^*

Convergence in $W^{1,p}_{loc}(\mathbb{T} \times \mathbb{R}^2, \mathbb{C})$. Up to a subsequence, we may assume, in view of Lemma 3.2 (and $|u_\varepsilon|_\infty \leq C_L$), that, for $1 \leq p < 3/2$,

$$u_\varepsilon \rightharpoonup u_* \qquad \text{in } W^{1,p}_{loc}(\mathbb{T} \times \mathbb{R}^2, \mathbb{C}) \text{ and a.e. as } \varepsilon \to 0.$$

Note also that outside C_{R_0}, since $|u_{\varepsilon,n}| \geq 1/2$ and $u_{\varepsilon,n}$ converges a.e. to u_ε, we have $|u_\varepsilon| \geq 1/2$ there, thus

$$|u_*(x)| \geq 1/2 \qquad \text{for } r \geq R_0.$$

We will show that $u_* = U_L^*$. Since u_ε satisfies (9), taking the exterior product of (9) with u_ε yields

$$\begin{cases} d^*(u_\varepsilon \times du_\varepsilon) & = & \frac{c_\varepsilon}{2} |\log \varepsilon| \partial_1 (1 - |u_\varepsilon|^2), \\ d(u_\varepsilon \times du_\varepsilon) & = & 2Ju_\varepsilon. \end{cases} \tag{51}$$

Passing to the weak limit in H^1_{loc} as $n \to +\infty$, we deduce from (20) that

$$\int_{\mathbb{T} \times \mathbb{R}^2} |\partial_1 u_\varepsilon|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{2\varepsilon^2} \leq C_L |\log \varepsilon|.$$

Thus $U_\varepsilon \in Y_\varepsilon$ and, if $\varepsilon \to 0$,

$$|\log \varepsilon| (1 - |u_\varepsilon|^2) \to 0 \qquad \text{in } L^2(\mathbb{T} \times \mathbb{R}^2),$$

so $|u_*| = 1$ a.e. and since $|c_\varepsilon| \leq K(L)$, as $\varepsilon \to 0$,

$$c_\varepsilon |\log \varepsilon| \partial_1 (1 - |u_\varepsilon|^2) \to 0 \qquad \text{in } H^{-1}(\mathbb{T} \times \mathbb{R}^2). \tag{52}$$

Concerning the second equation, we use (34) in Proposition 4 and the local jacobian estimate (29) to see that, in the distributional sense, as $\varepsilon \to 0$, up to a translation in the x_1 variable,

$$Ju_\varepsilon \to 2\pi \vec{\mathcal{H}}_L.$$

Passing to the limit in (51), we obtain

$$\begin{cases} \operatorname{div}(u_* \times \nabla u_*) &= 0, \\ \operatorname{curl}(u_* \times \nabla u_*) &= 2\pi \vec{\mathcal{H}}_L. \end{cases} \tag{53}$$

In order to identify u_*, we note that $u_* \times \nabla u_*$ is x_1-periodic and in L^p_{loc} $(1 \le p < 3/2)$ since

$$|u_*| = 1, \qquad \text{and} \qquad \nabla u_* \in L^p_{loc}.$$

Therefore, the vector fields $u_* \times \nabla u_*$ and $\vec{v}$ (defined in the introduction) both satisfy (53), except that $u_* \times \nabla u_*$ is, for the moment, only a L^p_{loc} map (for $1 \le p < 3/2$). Moreover, $u_* \times \nabla u_*$ satisfies

$$u_* \times \nabla u_* - \frac{\vec{e}_\theta}{r} \in L^2(\mathbb{T} \times \{r \ge R_0\})$$

by passing to the limit in (39), and by Lemma 1,

$$\vec{v} - \frac{\vec{e}_\theta}{r} \in L^2(\mathbb{T} \times \{r \ge R_0\}).$$

Furthermore, it is easily seen that $\vec{v} \in L^p_{loc}(\mathbb{T} \times \mathbb{R}^2)$ for $1 \le p < 3/2$. As a consequence, χ denoting a smooth function with support in C_{R_0+1} such that $\chi = 1$ in C_{R_0}, we may write

$$u_* \times \nabla u_* - \vec{v} = \chi(u_* \times \nabla u_* - \vec{v}) + (1-\chi)(u_* \times \nabla u_* - \vec{v}) \in (L^p_c + L^2)(\mathbb{T} \times \mathbb{R}^2) \tag{54}$$

for $1 \le p < 3/2$ and satisfies

$$\operatorname{div}(u_* \times \nabla u_* - \vec{v}) = 0 \qquad \text{and} \qquad \operatorname{curl}(u_* \times \nabla u_* - \vec{v}) = 0, \tag{55}$$

from which we infer $u_* \times \nabla u_* - \vec{v} \equiv 0$, that is $u_* \times \nabla u_* = \vec{v}$ and thus $u_* = U^*_L$ (up to a constant phase). Indeed, by (54), one may perform a Hodge-de Rham decomposition :

$$u_* \times \nabla u_* - \vec{v} = d\varphi + d^*\psi + \alpha\, dx_1 = \nabla\varphi + \operatorname{curl}\vec{\psi} + \alpha\vec{e}_1 \qquad \text{in } \mathbb{T} \times \mathbb{R}^2,$$

with $\alpha \in \mathbb{R}$, $d\psi = \operatorname{div}\vec{\psi} = 0$, and φ (resp. ψ) writes $\tilde{\varphi} + \hat{\varphi}$ (resp. $\tilde{\psi} + \hat{\psi}$) with $\tilde{\varphi}, \tilde{\psi} \in H^1$ and $\hat{\varphi}, \hat{\psi} \in W^{1,p}$ such that $\hat{\varphi}, \hat{\psi}$ are $\mathcal{O}(r^{-2})$ as $r \to +\infty$. By (55), we deduce that φ and ψ are harmonic and thus vanish in view of their behavior at infinity. Therefore, $u_* \times \nabla u_* - \vec{v} = \alpha\, dx_1$. Moreover, from the proof of Lemma 1, we know that, as $r \to +\infty$, $\vec{v} \cdot \vec{e}_1 = \mathcal{O}(r^{-2})$. Finally, passing to the limit in Lemmas 3.3 and 3.4, we deduce that $u_* = \exp(i\theta + i\varphi_*)$ for a smooth and x_1-periodic φ_*, thus

$$2\pi\alpha = \int_{\mathbb{T}} u_* \times \partial_1 u_* - \vec{v} \cdot \vec{e}_1\, dx_1 = \int_{\mathbb{T}} \partial_1 \varphi_* + \mathcal{O}(r^{-2}) = \mathcal{O}(r^{-2}) \to 0 \ \text{ as } r \to +\infty,$$

that is $\alpha = 0$. In view of the uniqueness of the possible weak limit, we have in $W^{1,p}_{loc}$,

$$u_{\varepsilon_j} \rightharpoonup U^*_L \qquad \text{as } \ j \to +\infty$$

for any sequence $\varepsilon_j \to 0$. We turn now to strong convergence outside $\mathcal{H}_L$.

Convergence in $\mathcal{C}^k_{loc}(\mathbb{T} \times \mathbb{R}^2 \setminus \mathcal{H}_L)$. The weak $W^{1,p}_{loc}$ $(1 \le p < 3/2)$ convergence implies in particular (up to a phase for u_ε)

$$u_\varepsilon \to U^*_L \qquad \text{in } L^1_{loc} \qquad \text{as } \varepsilon \to 0.$$

Moreover, by Lemma 3.3, u_ε is bounded in $\mathcal{C}^k_{loc}(\mathbb{T} \times \mathbb{R}^2 \setminus \mathcal{H}_L)$, thus, as $\varepsilon \to 0$,

$$u_\varepsilon \to U^*_L \qquad \text{in } \mathcal{C}^k_{loc}(\mathbb{T} \times \mathbb{R}^2 \setminus \mathcal{H}_L)$$

for all $k \in \mathbb{N}$, which is (6) in Theorem 1.

Assertion (5) ($|u_\varepsilon(x)| \to 1$ as $r \to +\infty$) in Theorem 1 is easily deduced from the fact that u_ε is lipschitz (for instance, $|\nabla u_\varepsilon|_\infty \le C_L/\varepsilon$) and $\int_{\mathbb{T} \times \mathbb{R}^2} (1 - |u_\varepsilon|^2)^2 < \infty$. We complete the proof of Theorem 1 with the following decay result.

Proposition 5. *We may write, for $r \geq R_0$, ε sufficiently small and $n \geq \exp(1/\varepsilon)$,*

$$u_{\varepsilon,n}(x) := \rho e^{i\varphi(x)+i\theta},$$

for φ a smooth real-valued function unique up to a multiple of 2π and $\rho \geq 1/2$. There exists constants $C_L > 0$ and $\lambda = \lambda(L)$, independent of $n \geq \exp(1/\varepsilon)$, such that, for $R_0 \leq R < (1-\gamma)n$,

$$\int_{\Omega_n(b)\setminus C_R} |\nabla\rho|^2 + \rho^2|\nabla\varphi|^2 + \frac{(1-\rho^2)^2}{2\varepsilon^2} \leq \frac{C_L}{R^\lambda} + \sigma_n(\varepsilon), \tag{56}$$

where $\gamma \in (0,1)$ is the one of lemma 3.1, and for $R_0 \leq R_1 \leq R_2 < (1-\gamma)n$,

$$\Big| E_\varepsilon(u_{\varepsilon,n}, \Omega_n(b) \cap (C_{R_2} \setminus C_{R_1})) - 2\pi^2 \log\Big(\frac{R_2}{R_1}\Big)\Big| \leq \frac{C_L}{R_1^\lambda} + 2\sigma_n(\varepsilon), \tag{57}$$

where $\sigma_n(\varepsilon)$ depends only on n and ε and $\sigma_n(\varepsilon) \to 0$ as $n \to +\infty$. In particular,

$$\lim_{n\to+\infty} p(u_{\varepsilon,n}) = p(u_\varepsilon) = 2\pi^2 L^2. \tag{58}$$

Note that (12) in Proposition 1 is deduced from (57) in Proposition 5 by passing to the limit as $n \to +\infty$ and (58) concludes the proof of (4) in Theorem 1. The asymptotic (13) of the energy on $\mathbb{T} \times C_{L+1}$ stated in Proposition 1 is a direct consequence of (31) in Proposition 4 and the strong convergence (for $L+1 \leq r \leq R_0$ if necessary) given in Lemma 3.3. Proposition 1 is thus a consequence of Proposition 5.

3.4 Proof of Proposition 5

The proof of Proposition 5 is based on the following decay lemma.

Lemma 3.5. *There exists a constant $C_L > 0$ such that, for every $R_0 \leq R < (1-\gamma)n$, $n \geq \exp(1/\varepsilon)$ and $0 < \varepsilon < \varepsilon_0(L)$ sufficiently small,*

$$\int_{\Omega_n(b)\setminus C_R} f_\varepsilon \leq C_L R \int_{\Omega_n(b)\cap\partial C_R} f_\varepsilon + \frac{C_L \varepsilon}{R^2} + \frac{1}{2}\sigma_n(\varepsilon),$$

where

$$f_\varepsilon := \frac{1}{2}\Big(\rho^2|\nabla\varphi|^2 + |\nabla\rho|^2 + \frac{(1-\rho^2)^2}{2\varepsilon^2}\Big)$$

and $\sigma_n(\varepsilon)$ depends only on ε, n and L and, for fixed ε, $\sigma_n \to 0$ as $n \to +\infty$.

Proof of Lemma 3.5. We argue as in [BOS] (Lemma 5.1). Since $|u_{\varepsilon,n}| \geq 1/2$ for $r \geq R_0$ and $u_{\varepsilon,n}$ has a degree one outside C_{R_0}, we may write, for $r \geq R_0$,

$$u_{\varepsilon,n} = \rho e^{i\varphi+i\theta},$$

where φ is a smooth real-valued function on $\Omega_n(b) \setminus C_{R_0}$ and $1/2 \leq \rho \leq C_L$. Equation (9) reads now for $r \geq R_0$ (with $\vec{e}_\theta = (0, -\sin\theta, \cos\theta)$),

$$-\Delta\rho + \rho\big|\nabla\varphi + \frac{\vec{e}_\theta}{r}\big|^2 - c_{\varepsilon,n}|\log\varepsilon|\rho\partial_1\varphi = \frac{1}{\varepsilon^2}\rho(1-\rho^2). \tag{59}$$

The estimate for the modulus is very close to the one in [BOS], whereas the estimate for the phase is slightly different because of the degree one at infinity.

Estimate for the modulus. Multiplying (59) by ρ^2-1 and integrating over $\Omega_n(b)\setminus C_R$ gives

$$\int_{\Omega_n(b)\setminus C_R} 2\rho|\nabla\rho|^2+\rho\frac{(1-\rho^2)^2}{\varepsilon^2}=\int_{\partial(\Omega_n(b)\setminus C_R)}\frac{\partial\rho}{\partial\nu}(1-\rho^2)+\int_{\Omega_n(b)\setminus C_R}\rho(1-\rho^2)\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2$$
$$-c_{\varepsilon,n}|\log\varepsilon|\int_{\Omega_n(b)\setminus C_R}\rho(1-\rho^2)\partial_1\varphi. \tag{60}$$

By Cauchy-Schwarz, since $\rho=1$ on $\partial\Omega_n(b)$,

$$\Big|\int_{\partial(\Omega_n(b)\setminus C_R)}\frac{\partial\rho}{\partial\nu}(1-\rho^2)\Big|=\Big|\int_{\Omega_n(b)\cap\partial C_R}\frac{\partial\rho}{\partial\nu}(1-\rho^2)\Big|\leq 2\varepsilon\int_{\Omega_n(b)\cap\partial C_R} f_\varepsilon. \tag{61}$$

From Lemma 3.3, we know that $|\nabla\varphi+\frac{\vec{e}_\theta}{r}|\leq C_L$ for $r\geq R_0$, thus $|\nabla\varphi|\leq C_L$ and then

$$\Big|\int_{\Omega_n(b)\setminus C_R}\rho(1-\rho^2)\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2\Big|\leq C_L\varepsilon\int_{\Omega_n(b)\setminus C_R}\frac{(1-\rho^2)^2}{2\varepsilon^2}+|\nabla\varphi|^2+\frac{1}{r^4}$$
$$\leq C_L\varepsilon\int_{\Omega_n(b)\setminus C_R} f_\varepsilon+\frac{C_L\varepsilon}{R^2}, \tag{62}$$

since $\int_{\{\mathbb{T}\times\mathbb{R}^2\setminus C_R\}} r^{-4}\leq CR^{-2}$. For the last term, Cauchy-Schwarz yields

$$|c_{\varepsilon,n}|\cdot|\log\varepsilon|\cdot\Big|\int_{\Omega_n(b)\setminus C_R}\rho(1-\rho^2)\partial_1\varphi\Big|\leq K(L)\varepsilon|\log\varepsilon|\int_{\Omega_n(b)\setminus C_R} f_\varepsilon. \tag{63}$$

Combining (61), (62) and (63) with (60), we deduce, since $\rho\geq 1/2$,

$$\int_{\Omega_n(b)\setminus C_R}|\nabla\rho|^2+\frac{(1-\rho^2)^2}{2\varepsilon^2}\leq C_LR\int_{\Omega_n(b)\cap\partial C_R} f_\varepsilon+r(\varepsilon)\int_{\Omega_n(b)\setminus C_R} f_\varepsilon+\frac{C_L\varepsilon}{R^2}. \tag{64}$$

Estimate for the phase. Concerning the phase, we will argue as for the proof of Lemma 3.4. Nevertheless, we need a more precise estimate for $\|b\|$, ensuring us that the helix is nearly centered around the x_1 axis, so that the phase φ is nearly 0 on $\partial\Omega_n(b)$. The result is given in the following lemma, whose proof is postponed to subsection 3.5.

Lemma 3.6. *We have, for fixed ε,*

$$\lim_{n\to+\infty}\frac{\|b\|}{n}=0.$$

For the proof of the estimate for the phase, we follow the lines of the proof of Lemma 3.4 and then consider, on $V_n:=\Omega_n(b)\setminus C_R$, for $R_0\leq r\leq R<(1-\gamma)n$,

$$v=e^{-i\theta}u_{\varepsilon,n}=\rho e^{i\varphi}.$$

We note that, by Lemma 3.1, for $R<(1-\gamma)n$, $\bar{C}_R\subset\Omega_n(b)$. We then perform a Hodge-de Rham decomposition of $v\times dv$ on V_n

$$v\times dv=\alpha dx_1+d\phi+d^*\psi, \tag{65}$$

where ϕ is a smooth function such that $\phi=0$ on ∂V_n, $\alpha\in\mathbb{R}$ is a constant and ψ is a 2-form such that $d\psi=0$ and $\psi_\top=0$ on ∂V_n. Applying the operators d and d^* to (65) and using the equation (40) for the phase, we deduce the equations in V_n

$$-\Delta\phi=-\frac{c_{\varepsilon,n}}{2}|\log\varepsilon|\partial_1(\rho^2-1)-\frac{\partial_\theta(\rho^2-1)}{r^2}, \tag{66}$$
$$-\Delta\psi=2Jv. \tag{67}$$

We now turn to estimates for ϕ, ψ and α. For $R \leq (1-\gamma)n$, we still have

$$|V_n| \geq 2\pi(\pi n^2 - \pi R^2) \geq 2\pi^2(1-(1-\gamma)^2)n^2 \geq \frac{n^2}{C_L}$$

and the estimate for α follows as for (44)

$$|\alpha| \leq \frac{r(\varepsilon)}{|V_n|} \leq \frac{C_L}{n^2}. \tag{68}$$

To estimate ϕ, we have as for (46)

$$\int_{V_n} |\nabla\phi|^2 \leq \varepsilon(K(L)|\log\varepsilon| + 1)\Big(\int_{V_n} \frac{(\rho^2-1)^2}{\varepsilon^2}\Big)^{1/2}\Big(\int_{V_n} |\nabla\phi|^2\Big)^{1/2},$$

and therefore

$$\int_{V_n} |\nabla\phi|^2 \leq r(\varepsilon)\int_{V_n} \frac{(\rho^2-1)^2}{\varepsilon^2} \leq r(\varepsilon)\int_{\Omega_n(b)\setminus C_R} f_\varepsilon. \tag{69}$$

We finally estimate ψ. As for the estimate (47), we write $\psi = \psi_0 + \psi_1$, where $\tilde{v} = v/|v|$,

$$\begin{cases} -\Delta\psi_0 = & 2J\tilde{v} = 0 & \text{in } V_n, \\ (\psi_0)_\top = & 0 & \text{on } \partial V_n, \\ (d^*\psi_0)_\top = & (v\times dv)_\top - \alpha dx_1 & \text{on } \partial V_n \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\psi_1 = & 2(Jv - J\tilde{v}) & \text{in } V_n, \\ (\psi_1)_\top = & 0 & \text{on } \partial V_n, \\ (d^*\psi_1)_\top = & 0 & \text{on } \partial V_n. \end{cases}$$

The estimate for ψ_1 follows as for (50)

$$\int_{V_n} |\nabla\psi_1|^2 \leq r(\varepsilon)\int_{V_n} |v\times dv|^2 \leq r(\varepsilon)\int_{V_n} f_\varepsilon. \tag{70}$$

Concerning ψ_0, we still have, as for (48),

$$\int_{V_n} |\nabla\psi_0|^2 \leq K\Big(n\|(v\times dv)_\top - \alpha dx_1\|^2_{L^2(\partial\Omega_n(b))} + R\|(v\times dv)_\top - \alpha dx_1\|^2_{L^2(\partial C_R)}\Big).$$

Since $R \leq (1-\gamma)n$, we deduce from (68) that

$$n\|\alpha dx_1\|^2_{L^2(\partial\Omega_n(b))} + R\|\alpha dx_1\|^2_{L^2(\partial C_R)} \leq \frac{C_L}{n^2},$$

and there holds

$$R\|(v\times dv)_\top\|^2_{L^2(\partial C_R)} \leq C_L R\int_{\partial C_R} f_\varepsilon.$$

It remains to estimate $n\|(v\times dv)_\top\|^2_{L^2(\partial\Omega_n(b))}$. To that aim, we note that on $\partial\Omega_n(b)$,

$$v(x) = e^{-i\theta}\frac{z-\beta_n}{|z-\beta_n|} = e^{-i\theta}(z-\beta_n),$$

with $z = (x_2 + ix_3)/n \in \partial D_1(\beta_n)$ and $\beta_n := b/n$. By scaling, we then have

$$n\|(v\times dv)_\top\|^2_{L^2(\partial\Omega_n(b))} = \int_{\partial D_1(\beta_n)} \Big|\nabla\Big(e^{-i\theta}(z-\beta_n)\Big)\Big|^2\, dz.$$

We use Lemma 3.6 to deduce that this last integral tends to 0 as $n \to +\infty$. Indeed,

$$\Big|\nabla\Big(e^{-i\theta}(z-\beta_n)\Big)\Big|^2 = \frac{1}{r^2} + 1 - 2\frac{\cos(\theta-\omega)}{r}, \tag{71}$$

where $(1,\omega)$ are the polar coordinates of $z-\beta_n$ (that is $z-\beta_n=e^{i\omega}$). By Lemma 3.6, as $n\to+\infty$, $\beta_n\to 0$, so $r\to 1$ and $\theta-\omega\to 0$ pointwise and the right-hand side of (71) is uniformly bounded (since $r\geq 1-\gamma$), thus by dominated convergence,

$$\tilde{\sigma}_n(\varepsilon):=n\|(v\times dv)_\top\|^2_{L^2(\partial\Omega_n(b))}\to 0 \qquad \text{as } n\to+\infty.$$

As a consequence, we have

$$\int_{V_n}|\nabla\psi_0|^2\leq K\tilde{\sigma}_n(\varepsilon)+C_LR\int_{\partial C_R}f_\varepsilon+\frac{C_L}{n^2}. \tag{72}$$

From (70) and (72), we infer

$$\int_{V_n}|\nabla\psi|^2\leq C_LR\int_{\partial C_R}f_\varepsilon+r(\varepsilon)\int_{V_n}f_\varepsilon+K\tilde{\sigma}_n(\varepsilon)+\frac{C_L}{n^2}. \tag{73}$$

Finally, combining (68), (69) and (73), we obtain

$$\int_{V_n}|\rho\nabla\varphi|^2\leq C_L\int_{V_n}|v\times dv|^2\leq C_LR\int_{\partial C_R}f_\varepsilon+r(\varepsilon)\int_{V_n}f_\varepsilon+\frac{1}{2}\sigma_n(\varepsilon), \tag{74}$$

where $\sigma_n(\varepsilon)\to 0$ as $n\to+\infty$. From (64) and (74), we conclude, for $R_0\leq R<(1-\gamma)n$,

$$\int_{\Omega_n(b)\setminus C_R}f_\varepsilon\leq C_LR\int_{\partial C_R}f_\varepsilon+r(\varepsilon)\int_{\Omega_n(b)\setminus C_R}f_\varepsilon+\frac{C_L\varepsilon}{R^2}+\frac{1}{2}\sigma_n(\varepsilon).$$

Taking $0<\varepsilon<\varepsilon_0(L)$ sufficiently small (so that $r(\varepsilon)\leq 1/2$), we are led to the conclusion. □

Proof of Proposition 5. Consider the function, for $R_0\leq R<(1-\gamma)n$,

$$g_n(R):=\int_{\Omega_n(b)\setminus C_R}f_\varepsilon.$$

From Lemma 3.5, we deduce that g_n satisfies

$$g_n(R)\leq C_LR\int_{\partial C_R}f_\varepsilon+\frac{C_L\varepsilon}{R^2}+\frac{1}{2}\sigma_n(\varepsilon)=-C_LRg_n'(R)+\frac{C_L}{R^2}+\frac{1}{2}\sigma_n(\varepsilon). \tag{75}$$

Therefore, we have, with $\lambda:=C_L^{-1}>0$,

$$\frac{d}{dR}\Big(R^\lambda g_n(R)\Big)=\lambda R^{\lambda-1}\Big(g_n(R)+C_LRg_n'(R)\Big)\leq\lambda R^{\lambda-1}\Big(\frac{C_L}{R^2}+\frac{1}{2}\sigma_n(\varepsilon)\Big).$$

Enlarging C_L if necessary, we may assume $C_L\geq 1$, so $\lambda\leq 1$. Integrating between R_0 and R yields

$$R^\lambda g_n(R)-R_0^\lambda g_n(R_0)\leq\frac{1}{2}\sigma_n(\varepsilon)(R^\lambda-R_0^\lambda)+C_L\frac{\lambda}{\lambda-2}\Big(R^{\lambda-2}-R_0^{\lambda-2}\Big).$$

Moreover, we have by Lemma 3.3

$$g_n'(R_0)\leq C_L,$$

thus applying (75) with $R=R_0$, we obtain, for n sufficiently large, $g_n(R_0)\leq C_L$, and therefore

$$g_n(R)\leq\frac{C_L}{R^\lambda}+\frac{1}{2}\sigma_n(\varepsilon),$$

which concludes the proof of (56). Concerning (57), it suffices to write

$$E_\varepsilon(u_{\varepsilon,n},C_{R_2}\setminus C_{R_1})=\int_{R_1\leq r\leq R_2}f_\varepsilon+\frac{1}{2}\int_{R_1\leq r\leq R_2}\rho^2(|\nabla\varphi+\frac{\vec{e}_\theta}{r}|^2-|\nabla\varphi|^2).$$

Since $C_{R_1} \subset \bar{C}_{R_2} \subset \Omega_n(b)$ for $R_1 < R_2 < (1-\gamma)n$, we have by smoothness of φ,

$$\int_{R_1 \le r \le R_2} \frac{\partial_\theta \varphi}{r^2} = 0,$$

thus

$$\begin{aligned}
\Big| E_\varepsilon(u_{\varepsilon,n}, C_{R_2} \setminus C_{R_1}) - 2\pi^2 \log\Big(\frac{R_2}{R_1}\Big)\Big| &\le |g_n(R_2) - g_n(R_1)| + \frac{1}{2}\Big| \int_{R_1 \le r \le R_2} (\rho^2 - 1)\Big(\frac{1}{r^2} + 2\frac{\partial_\theta \varphi}{r^2}\Big)\Big| \\
&\le |g_n(R_2)| + |g_n(R_1)| + \frac{1}{2}\int_{R_1 \le r \le R_2} |\rho^2 - 1|\Big(\frac{1}{r^2} + 2\frac{|\partial_\theta \varphi|}{r^2}\Big) \\
&\le \frac{C_L}{R_1^\lambda} + 2\sigma_n(\varepsilon) + \frac{C\varepsilon}{R_1}\int_{R_1 \le r \le R_2} \frac{(\rho^2-1)^2}{2\varepsilon^2} + \frac{1}{r^3} + |\nabla \varphi|^2 \\
&\le \frac{C_L}{R_1^\lambda} + 2\sigma_n(\varepsilon) + \frac{C_L \varepsilon |\log \varepsilon|}{R_1} \le \frac{C_L}{R_1^\lambda} + 2\sigma_n(\varepsilon),
\end{aligned}$$

which is (57). We easily deduce from this decay that

$$p(u_\varepsilon) = 2\pi^2 L^2. \tag{76}$$

Indeed, let $R \ge R_0$ and fix χ a smooth function compactly supported such that $0 \le \chi \le 1$, $\chi = 1$ on $C_R(0)$, and $\chi = 0$ outside $C_{2R}(0)$. We can choose χ radial. Recalling the definition of the momentum (8), we have (we already know that $U_\varepsilon \in Y_\varepsilon$)

$$p(u_\varepsilon) = \int_{\mathbb{T}\times\mathbb{R}^2} (iu_\varepsilon, \partial_1 u_\varepsilon)\chi + \int_{\mathbb{T}\times\mathbb{R}^2} (1-\chi)(\rho_\varepsilon^2 - 1)\partial_1 \varphi_\varepsilon, \tag{77}$$

since the last term in (8) vanishes if χ is radial. On the other hand, for $n \ge \exp(1/\varepsilon)$, since, as already seen, $\varphi_{\varepsilon,n}$ is periodic in the x_1 variable,

$$p(u_{\varepsilon,n}) = \int_{\mathbb{T}\times\mathbb{R}^2} (iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n})\chi + \int_{\mathbb{T}\times\mathbb{R}^2} (1-\chi)(\rho_{\varepsilon,n}^2 - 1)\partial_1 \varphi_{\varepsilon,n}. \tag{78}$$

By strong H^1_{loc} convergence as $n \to +\infty$, the first term in (78) converges to the first term in (77). For the second terms in (77) and (78), they both have the decay established in Propositions 1 and 5, thus, for any $R_0 < R < (1-\gamma)n$ and any $n \ge \exp(1/\varepsilon)$,

$$\Big| \int_{\Omega_n(b)} (1-\chi)(\rho_{\varepsilon,n}^2 - 1)\partial_1 \varphi_{\varepsilon,n}\Big| + \Big| \int_{\mathbb{T}\times\mathbb{R}^2} (1-\chi)(\rho_\varepsilon^2 - 1)\partial_1 \varphi_\varepsilon\Big| \le \frac{C_L}{R^\lambda} + \sigma_n(\varepsilon).$$

Next, let $n \to +\infty$ to deduce

$$\limsup_{n\to+\infty} |p(u_{\varepsilon,n}) - p(u_\varepsilon)| \le \frac{C_L}{R^\lambda},$$

and then let $R \to +\infty$. This proves (76), which is the assertion (4) in Theorem 1 for the momentum, and thus completes the proof of Theorem 1. □

3.5 Proof of Lemma 3.6

The proof of Lemma 3.6 relies on the reduction to a 2-dimensional problem, for which results about the location of the vortices can be proved, with the help of the renormalized energy (see [BBH2]): the limiting vortices are critical points of the renormalized energy. We consider the map $w_n : \mathbb{T} \times D_1 \to \mathbb{C}$ defined by

$$w_n(x) := u_{\varepsilon,n}(x_1, nx_2, nx_3)$$

with ε fixed and let $n \to +\infty$. By scaling from (19) and (20), we have, with $\delta := \varepsilon/n$,

$$\frac{1}{2}\int_{\mathbb{T}\times D_1} |\nabla_{2,3}w_n|^2 + \frac{(1-|w_n|^2)^2}{2\delta^2} + \frac{n^2}{2}\int_{\mathbb{T}\times D_1} |\partial_1 w_n|^2 \le 2\pi^2 \log n + C_L|\log\varepsilon|, \tag{79}$$

$$\frac{1}{2}\int_{\mathbb{T}\times D_1} \frac{(1-|w_n|^2)^2}{2\delta^2} + \frac{n^2}{2}\int_{\mathbb{T}\times D_1} |\partial_1 w_n|^2 \le C_L|\log\varepsilon|, \tag{80}$$

$$w_n = e^{i\theta} \qquad \text{on } \mathbb{T}\times\partial D_1 \tag{81}$$

and, with $\Delta_{2,3} := \partial_2^2 + \partial_3^2$,

$$\Delta_{2,3}w_n + \frac{w_n}{\delta^2}(1-|w_n|^2) = ic_{\varepsilon,n}|\log\varepsilon|n^2\partial_1 w_n - n^2\partial_1^2 w_n. \tag{82}$$

Here, we adopt the point of view $\varepsilon > 0$ fixed and $n \to +\infty$, that is $\delta \to 0$. We expect that, as $n \to +\infty$, w_n tends to a map independent of the variable x_1, with only one vortex at $\beta_n := b/n$ (the bound (79) is then the natural one for w_n to have only one vortex) and merely satisfies the 2-dimensional Ginzburg-Landau equation (82) (provided the right-hand side of (82) is small in some sense), so that we expect that the limiting vortex must be a critical point of the renormalized energy, which is 0. The proof is divided in several steps, and we prove all the ingredients needed in the proof of Theorem VII.4 in [BBH2]. In the sequel, K will denote a constant independent of n, but depending only on ε and L.

Step 1: $W^{1,p}$ bounds for w_n. We prove that, for any $1 \le p < 3/2$,

$$\int_{\mathbb{T}\times D_1} |\nabla w_n|^p \le K_p. \tag{83}$$

We proceed as in the proof of Proposition C.2 in Appendix C in [BOS].

Estimate for the modulus. Since w_n satisfies (82), then $\rho := |w_n|$ satisfies

$$-\Delta_{2,3}\rho^2 - n^2\partial_1\rho^2 + 2|\nabla_{2,3}w_n|^2 + 2n^2|\partial_1 w_n|^2 = 2\frac{\rho^2}{\delta^2}(1-\rho^2) - c_{\varepsilon,n}n^2|\log\varepsilon|(w_n, i\partial_1 w_n). \tag{84}$$

We consider $\bar\rho := \max(\rho, 1-\delta^{1/2})$. Since $\rho = 1$ on $\mathbb{T}\times\partial D_1$, then $\bar\rho^2 - 1 = 0$ on $\mathbb{T}\times\partial D_1$ and multiplying (84) by $\bar\rho^2 - 1$ and integrating yields

$$\begin{aligned}\int_{\mathbb{T}\times D_1} (\nabla_{2,3}\rho^2)\cdot(\nabla_{2,3}\bar\rho^2) + n^2(\partial_1\rho)(\partial_1\bar\rho) + \frac{2}{\delta^2}\int_{\mathbb{T}\times D_1}(1-\rho^2)(1-\bar\rho^2) \\ = \int_{\mathbb{T}\times D_1} 2(1-\bar\rho^2)(|\nabla_{2,3}w_n|^2 + n^2|\partial_1 w_n|^2) + c_{\varepsilon,n}n^2|\log\varepsilon|\int_{\mathbb{T}\times D_1}(1-\bar\rho^2)(iw_n, \partial_1 w_n).\end{aligned} \tag{85}$$

We note that the integrand in the second integral of the left hand side is non-negative, since either $\rho \ge 1-\delta^{1/2}$ and then $\rho = \bar\rho$ so $(1-\rho^2)(1-\bar\rho^2) = (1-\rho^2)^2 \ge 0$; either $0 \le \rho \le 1-\delta^{1/2} \le 1$ and then $\rho, \bar\rho \in [0,1]$ so $(1-\rho^2), (1-\bar\rho^2) \ge 0$. Moreover, $0 \le 1-\bar\rho \le \delta^{1/2}$ by construction, so $0 \le 1-\bar\rho^2 \le 2\delta^{1/2}$ and then, by (79),

$$\int_{\mathbb{T}\times D_1} 2(1-\bar\rho^2)(|\nabla_{2,3}w_n|^2 + n^2|\partial_1 w_n|^2) \le 4\delta^{1/2}(4\pi^2\log n + K) \le K, \tag{86}$$

since $\delta^{1/2} \le Kn^{-1/2}$. Finally, we carefully estimate the last term in (85). First, note that, by (80),

$$|\{\rho < 1-\delta^{1/2}\}| \le K\delta.$$

As a consequence, by Cauchy-Schwarz and (80),

$$\begin{aligned}\Big|c_{\varepsilon,n}n^2|\log\varepsilon|\int_{\{\rho<1-\delta^{1/2}\}}(1-\bar\rho^2)(iw_n,\partial_1 w_n)\Big| &\le K(L)\delta^{1/2}|\log\varepsilon|n^2\int_{\{\rho<1-\delta^{1/2}\}}|\partial_1 w_n|\\ &\le K\delta^{1/2}n^2\Big(\frac{K}{n^2}\Big)^{1/2}(K\delta)^{1/2}\le K,\end{aligned}$$

since $\delta \le Kn^{-1}$. Moreover, since $\rho=\bar\rho$ in $\{\rho\ge 1-\delta^{1/2}\}$, by (80),

$$\begin{aligned}\Big|c_{\varepsilon,n}n^2|\log\varepsilon|\int_{\{\rho\ge 1-\delta^{1/2}\}}(1-\bar\rho^2)(iw_n,\partial_1 w_n)\Big| &\le K(L)|\log\varepsilon|\int_{\mathbb{T}\times D_1}(n|1-\rho^2|)(n|\partial_1 w_n|)\\ &\le K\int_{\mathbb{T}\times D_1}\frac{(1-\rho^2)^2}{\delta^2}+n^2|\partial_1 w_n|^2\le K.\end{aligned}$$

Therefore, the last term in (85) verifies

$$\Big|c_{\varepsilon,n}n^2|\log\varepsilon|\int_{\mathbb{T}\times D_1}(1-\bar\rho^2)(iw_n,\partial_1 w_n)\Big|\le K. \tag{87}$$

Finally, $\nabla(\bar\rho^2)=\nabla(\rho^2)$ if $\rho\ge 1-\delta^{1/2}$ and 0 otherwise, so, inserting (86) and (87) into (85) yields

$$\int_{\{\rho\ge 1-\delta^{1/2}\}}|\nabla\rho^2|^2\le K$$

and then, since $\rho\ge 1-\delta^{1/2}\ge 1/2$ if $\delta\le 1/4$, for $1\le p\le 2$,

$$\int_{\{\rho\ge 1-\delta^{1/2}\}}|\nabla\rho|^p\le K_p. \tag{88}$$

Since, as already seen, $|\{\rho<1-\delta^{1/2}\}|\le K\delta$, we infer by Hölder inequality that, for $1\le p<2$,

$$\int_{\{\rho<1-\delta^{1/2}\}}|\nabla\rho|^p\le|\{\rho<1-\delta^{1/2}\}|^{1-p/2}\Big(\int_{\mathbb{T}\times D_1}|\nabla\rho|^2\Big)^{p/2}\le Kn^{p/2-1}(\log n)^{p/2}\le K_p. \tag{89}$$

We deduce from (88) and (89) the estimate for the modulus, for $1\le p<2$,

$$\int_{\mathbb{T}\times D_1}|\nabla\rho|^p\le K_p. \tag{90}$$

Estimate for the pre Jacobian. We perform a Hodge-de Rham decomposition of $w_n\times dw_n$:

$$w_n\times dw_n=d\varphi+d^*\psi+\alpha dx_1, \tag{91}$$

where $\alpha\in\mathbb{R}$ is a constant, φ is a function satisfying $\varphi=0$ on $\mathbb{T}\times\partial D_1$, and ψ is a 2-form such that $\psi_\top=0$ on $\mathbb{T}\times\partial D_1$ and $d\psi=0$. To estimate α, we write

$$2\pi^2\alpha=\alpha|\mathbb{T}\times D_1|=\int_{\mathbb{T}\times D_1}\langle w_n\times dw_n,dx_1\rangle=\frac{1}{n^2}\int_{\mathbb{T}\times D_n}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})=\frac{2\pi^2L^2}{n^2}$$

by scaling and in view of the constraint on the momentum, so

$$\alpha=\frac{L^2}{n^2}. \tag{92}$$

Applying the d and the d^* operators to (91) and using (81) and (82), we deduce the equations

$$\left\{\begin{array}{rcll} -\Delta\psi &=& 2Jw_n & \text{in } \mathbb{T}\times D_1,\\ \psi_\top &=& 0 & \text{on } \mathbb{T}\times\partial D_1,\\ (d^*\psi)_\top &=& d\theta & \text{on } \mathbb{T}\times\partial D_1 \end{array}\right. \tag{93}$$

and

$$\left\{\begin{array}{rcll} -(\Delta_{2,3}+n^2\partial_1^2)\varphi &=& -\frac{c_{\varepsilon,n}}{2}|\log\varepsilon|n^2\partial_1(\rho^2-1) & \text{in } \mathbb{T}\times D_1,\\ \varphi &=& 0 & \text{on } \mathbb{T}\times\partial D_1. \end{array}\right. \tag{94}$$

From Proposition 3.1 in [BO] (since $|d\theta|_\infty = 1$), we infer from (93) that, for $1 \le p < 3/2$,

$$\int_{\mathbb{T}\times D_1} |\nabla\psi|^p \le K_p. \tag{95}$$

Multiplying (94) by φ and integrating by parts yields by Cauchy-Schwarz and (80)

$$\begin{aligned}\int_{\mathbb{T}\times D_1} |\nabla_{2,3}\varphi|^2 + n^2|\partial_1\varphi|^2 &= \frac{c_{\varepsilon,n}}{2}|\log\varepsilon| \int_{\mathbb{T}\times D_1} (n(\rho^2-1))(n\partial_1\varphi)\\ &\le K(L)\int_{\mathbb{T}\times D_1} \frac{(\rho^2-1)^2}{\delta^2} + n^2|\partial_1\varphi|^2 \le K.\end{aligned}$$

As a consequence, by Hölder inequality, for $1 \le p \le 2$,

$$\int_{\mathbb{T}\times D_1} |\nabla_{2,3}\varphi|^p + n^p|\partial_1\varphi|^p \le K. \tag{96}$$

Therefore, combining (92), (95), (96) with (91), we obtain, for $1 \le p < 3/2$,

$$\int_{\mathbb{T}\times D_1} |w_n \times dw_n|^p \le K_p. \tag{97}$$

To conclude, we use the identity

$$\rho_n^2|\nabla w_n|^2 = \rho_n^2|\nabla\rho_n|^2 + |w_n\times dw_n|^2,$$

and the estimate $|\nabla w_n|_\infty \le Kn$, which comes by scaling from (26), to deduce,

$$\begin{aligned}|\nabla w_n|^2 &= |\nabla\rho_n|^2 + |w_n\times dw_n|^2 + (1-|w_n|^2)(|\nabla w_n|^2 - |\nabla\rho_n|^2)\\ &\le |\nabla\rho_n|^2 + |w_n\times dw_n|^2 + Kn\big|1-|w_n|^2\big|\cdot|\nabla w_n|\\ &\le |\nabla\rho_n|^2 + |w_n\times dw_n|^2 + \frac{1}{2}|\nabla w_n|^2 + Kn^2(1-|w_n|^2)^2,\end{aligned}$$

thus

$$|\nabla w_n|^2 \le K\Big(|\nabla\rho_n|^2 + |w_n\times dw_n|^2 + n^2(1-|w_n|^2)^2\Big),$$

and then, for $1 \le p < 3/2$,

$$\int_{\mathbb{T}\times D_1} |\nabla w_n|^p \le K\int_{\mathbb{T}\times D_1} |\nabla\rho_n|^p + |w_n\times dw_n|^p + K\Big(\int_{\mathbb{T}\times D_1} \frac{(1-|w_n|^2)^2}{\delta^2}\Big)^{p/2}.$$

Estimate (83) follows then from (80), (90) and (97). $\square$

From Step 1, we know that, up to a subsequence, w_n weakly converges in $W^{1,p}$ to a map w_* in $W^{1,p}(\mathbb{T} \times D_1, \mathbb{S}^1)$ for $1 \le p < 3/2$, as $n \to +\infty$, satisfying $w_* = e^{i\theta}$ on $\mathbb{T} \times \partial D_1$. Moreover, from (80),

$$\int_{\mathbb{T}\times D_1} |\partial_1 w_n|^2 \le \frac{K}{n^2} \to 0,$$

thus w_* is independent of the variable x_1. We will denote $\tilde{w}_* = w_*(x_1, .)$ for any $x_1 \in \mathbb{T}$. We denote also $\beta_* = \lim_{n\to+\infty} b/n \in D_1$ (and not $\in \bar{D}_1$, since we already know that $\|b\| \le (1-\gamma)n$).

Step 2: The vector field $\tilde{w}_* \times \nabla_{2,3}\tilde{w}_*$ is divergence free. Let $\zeta \in \mathcal{C}_0^1(D_1, \mathbb{R})$. We write the right-hand side of (82) as $\partial_1 \Upsilon_n$, where $\Upsilon_n := ic_{\varepsilon,n} n^2 |\log\varepsilon| w_n - n^2 \partial_1 w_n$. Therefore, by (82),

$$\langle \mathrm{div}_{2,3}(w_n \times \nabla_{2,3} w_n), \zeta\rangle = \langle w_n \times \Delta_{2,3} w_n, \zeta\rangle = \langle \partial_1 \Upsilon_n, \zeta\rangle = 0,$$

since ζ does not depend on x_1 and Υ_n is x_1-periodic. As a consequence, passing to the limit as $n \to +\infty$ (up to the subsequence), we obtain that the vector field $\tilde{w}_* \times \nabla_{2,3}\tilde{w}_*$ is divergence free. $\square$

We then apply Remark I.1 in chapter I of [BBH2] to conclude from Steps 1 and 2 that

$$\tilde{w}_* = w_0 \exp(i\kappa \log|z - \beta_*|)\exp(i\chi), \tag{98}$$

where $z = x_2 + ix_3$, w_0 is the canonical harmonic map associated to the boundary map $e^{i\theta}$ and the singularity β_*, κ is a real constant and χ the solution of

$$\begin{cases} -\Delta\chi &= 0 \quad \text{in } D_1, \\ \chi + \kappa\log|z-\beta_*| &= 0 \quad \text{on } \partial D_1. \end{cases}$$

Step 3: Strong convergence outside $\mathbb{T} \times \{\beta_*\}$. We prove that, a ball $B_R(a)$ in $\mathbb{T} \times (\bar{D}_1 \setminus \{\beta_*\})$ being given, for n sufficiently large (depending on the ball), we have

$$\big||w_n| - 1\big| \le \frac{K}{n^2} \qquad \text{in } B_R(a),$$

$$\|\nabla_{2,3} w_n\|_{L^\infty(B_R(a))} + n^2\|\partial_1 w_n\|_{L^\infty(B_R(a))} + n^2\|\partial_1^2 w_n\|_{L^\infty(B_R(a))} \le K.$$

These estimates are similar to the bounds in $\mathcal{C}^k_{loc}(\mathbb{T}\times\mathbb{R}^2 \setminus \mathcal{H}_L)$ given in Lemma 3.3, and are also related to the result given in Theorem VI.1 in [BBH2]. We define

$$\hat{w}_n(x) := \Big(1 + \frac{c_{\varepsilon,n}^2}{4}\varepsilon^2|\log\varepsilon|^2\Big)^{-1/2} \exp\Big(-i\frac{c_{\varepsilon,n}}{2}|\log\varepsilon| x_1\Big) w_n(x)$$

in $\mathbb{T} \times D_1$, which verifies

$$(\Delta_{2,3} + n^2\partial_1^2)\hat{w}_n + \frac{\hat{w}_n}{\hat{\delta}^2}(1 - |\hat{w}_n|^2) = 0, \tag{99}$$

where $\hat{\delta}^2 := (1 + \frac{c_{\varepsilon,n}^2}{4}\varepsilon^2|\log\varepsilon|^2)^{-1}\delta^2$. We will follow the lines of the proof of Theorem IV.1 in [BBO]. We do not prove Step 1 there. However, from Lemma 3.3, we know that

$$\eta := \sup_{\mathbb{T}\times D_n \setminus C_{R_0}} \big||u_{\varepsilon,n}| - 1\big| \le C_L \varepsilon^2|\log\varepsilon| \le 1/2 \tag{100}$$

for $0 < \varepsilon < \varepsilon_0(L)$ sufficiently small. Let us fix $R \in (0,1)$ and $a \in \mathbb{T} \times (\bar{D}_1 \setminus \{\beta_*\})$, and denote $B_R(a)$ the ball *in* $\mathbb{T} \times \bar{D}_1$ of radius R centered at a. By (100), we have for n sufficiently large (depending on $B_R(a)$),

$$|\hat{w}_n| \ge \frac{1}{2} \qquad \text{in } \bar{B}_R(a),$$

so that we may write $\hat{w}_n = \rho_n e^{i\varphi_n}$ in $\bar{B}_R(a)$, for a φ_n such that

$$\frac{1}{|B_{7R/8}(a)|}\int_{B_{7R/8}(a)} \varphi_n \in [0, 2\pi).$$

In the proof of [BBO], we replace each time the standard Laplace operator Δ by $\Delta_{2,3} + n^2\partial_1^2$, so that the scaled energy now writes

$$\tilde{F}_{\hat{\delta}}(\hat{w}_n, a, r) = \frac{1}{2r}\int_{B_r(a)} |\nabla_{2,3}\hat{w}_n|^2 + n^2|\partial_1\hat{w}_n|^2 + \frac{(1-|\hat{w}_n|^2)^2}{2\hat{\delta}^2}.$$

We follow the lines of Step 2 of the proof of Theorem IV.1 in [BBO], which implies the existence of $n_0 = n_0(\varepsilon, L, R, a) \in \mathbb{N}$ such that, for $x \in B_{7R/8}(a)$, $n \geq n_0$, $\mu \in (0, 1/2)$ and $0 < r < R/8$, then

$$\tilde{F}_{\hat{\delta}}(a, \mu r) \leq K_0(\mu^2 + \mu^{-1}(n^{-1} + \eta))\tilde{F}_{\hat{\delta}}(a, r),$$

where K_0 is absolute. Note that we may here reach the boundary $\mathbb{T} \times \partial D_1$, but since the boundary map $e^{i\theta}$ is independent of n and smooth of modulus 1, this does not change the proof. In particular, for μ and $\varepsilon < \varepsilon_0$ sufficiently small (μ and ε_0 absolute) (note that $\eta \to 0$ as $\varepsilon \to 0$ uniformly in n by (100)), we have

$$\tilde{F}_{\hat{\delta}}(x, \mu r) \leq \frac{1}{2}\tilde{F}_{\hat{\delta}}(x, r).$$

Consequently, we infer from this decay and the $W^{1,p}$ bound (83) as in Step 3 in [BBO] that

$$\|\hat{w}_n\|_{\mathcal{C}^{0,\alpha}(B_{6R/8}(a))} \leq K,$$

for an $\alpha \in (0, 1)$ depending on μ, and K depends only on L, ε, and $B_R(a)$. In particular,

$$\|\rho_n\|_{\mathcal{C}^{0,\alpha}(B_{6R/8}(a))} \leq K.$$

The equation for the phase is then

$$\mathrm{div}_{2,3}(\rho_n^2\nabla_{2,3}\varphi_n) + n^2\partial_1(\rho_n^2\partial_1\varphi_n) = 0 \qquad \text{in } B_{6R/8}(a),$$

from which we infer by Schauder estimates

$$\|\varphi_n\|_{\mathcal{C}^{1,\alpha}(B_{5R/8}(a))} \leq K. \tag{101}$$

We finally have the estimate for $1 - \rho_n^2$

$$0 \leq 1 - \rho_n^2 \leq \frac{K}{n^2} \qquad \text{in } B_{R/2}(a). \tag{102}$$

The lower bound is usual for the Ginzburg-Landau equation (99), and here is also a consequence of Lemma 4. The upper one is derived as in Step 5 in [BBO]. The equation for $h_n := 1 - \rho_n^2$ is

$$-(\Delta_{2,3} + n^2\partial_1^2)h_n + \frac{\rho_n}{\hat{\delta}^2}(1 + \rho_n)h_n = \rho_n|\nabla\varphi_n|^2 \qquad \text{in } B_{7R/8}(a),$$

thus by (100) and (101),

$$-(\Delta_{2,3} + n^2\partial_1^2)h_n + \frac{1}{2\hat{\delta}^2}h_n \leq K \qquad \text{in } B_{5R/8}(a),$$

and $h_n = 0$ on $\mathbb{T} \times \partial D_1$. Therefore, as in Lemma 2 in [BBH1], we obtain (102). A bootstrap argument, as in [BBH1] and Step 6 in [BBO], shows that

$$n^2\|\partial_1^2 w_n\|_{L^\infty(B_{R/2}(a))} + n^2\|\partial_1 w_n\|_{L^\infty(B_{R/2}(a))} \leq K \tag{103}$$

and then $\tilde{w}_* \in \mathcal{C}^\infty(D_1 \setminus \{\beta_*\}) \cap \mathcal{C}^0(\bar{D}_1 \setminus \{\beta_*\})$. $\square$

Step 4: Convergence for the potential term. *Let*

$$W_n := \frac{(1-|w_n|^2)^2}{4\delta^2}.$$

Then (up to a subsequence), in the weak $\star$ topology of $\mathcal{C}(\bar{D}_1)$,

$$\int_{\mathbb{T}} W_n \, dx_1 \rightharpoonup W_* = m\delta_{\{\beta_*\}}, \qquad \textit{with } m \in \mathbb{R}_+.$$

This step is the analogue of Lemma VII.1 in chapter VII of [BBH2]. First, note that by (80), $\int_{\mathbb{T}} W_n \, dx_1$ is bounded in $L^1(D_1)$, thus we may assume $\int_{\mathbb{T}} W_n \, dx_1 \rightharpoonup W_*$ in the weak $\star$ topology of $\mathcal{C}(\bar{D}_1)$. It remains to establish the structure of the measure W_*, which will follow from the strong convergence results of Step 3. Indeed, (102) implies that for $B_R(a) \subset \mathbb{T} \times (D_1 \setminus \{\beta_*\})$, we have

$$W_n(B_{R/2}(a)) \leq \frac{K}{n^2} \to 0 \text{ as } n \to +\infty,$$

thus $W_*(B_{R/2}(a)) = 0$. The measure W_* is then nonnegative and has a support included in $\{\beta_*\}$: it is then of the form $m\delta_{\{\beta_*\}}$, with $m \in \mathbb{R}_+$. $\square$

Step 5: An auxiliary problem. Let

$$q_n := (\partial_2 w_n, \partial_1 \Upsilon_n) - i(\partial_3 w_n, \partial_1 \Upsilon_n),$$

that we extend by 0 outside $\mathbb{T} \times D_1$. There exists $\lambda = \lambda(\varepsilon) \in \mathbb{R}$ such that, for all $\phi \in \mathcal{C}^1(\bar{D}_1, \mathbb{C})$, as $n \to +\infty$,

$$\int_{\mathbb{T}\times D_1} q_n(x)\phi(x_2,x_3)\, dx = n^2 \int_{\mathbb{T}\times D_1} (c_{\varepsilon,n}|\log\varepsilon|(iw_n,\partial_1 w_n) - |\partial_1 w_n|^2)\frac{\partial\phi}{\partial\bar{z}} \to -\lambda\frac{\partial\phi}{\partial\bar{z}}(\beta_*), \tag{104}$$

where $2\frac{\partial}{\partial\bar{z}} = (\partial_2 - i\partial_3)$. In other words, the distribution in $\mathbb{R}^2$ $S_n : \phi \mapsto \int_{\mathbb{T}\times D_1} q_n(x)\phi(x_2,x_3)\, dx$ converges as a distribution to $\lambda\frac{\partial}{\partial\bar{z}}\delta_{\beta_*}$. Moreover, the distribution in $\mathbb{R}^2$

$$\Lambda_n := \frac{1}{2\pi}\log|(x_2,x_3)| * S_n$$

is bounded in $L^p(D_1)$, $1 \leq p < 2$ and converges in the sense of distributions to

$$\Lambda_* := \frac{\lambda}{2\pi}\frac{\partial}{\partial\bar{z}}\log|(x_2,x_3)| * \delta_{\beta_*}.$$

Let us first derive the first identity in (104) by integration by parts (ϕ does not depend on x_1)

$$\int_{\mathbb{T}\times D_1} (\partial_2 w_n, \partial_1^2 w_n)\phi(x_2,x_3)\, dx = -\int(\partial_2\partial_1 w_n, \partial_1 w_n)\phi = -\frac{1}{2}\int \partial_2(|\partial_1 w_n|^2)\phi = \frac{1}{2}\int |\partial_1 w_n|^2\partial_2\phi,$$

where we have used that $\partial_1 w_n = 0$ on $\mathbb{T}\times\partial D_1$. Similarly, since $2(\partial_2 w_n, i\partial_1 w_n) = 2\partial_1 w_n \times \partial_2 w_n = \partial_1(w_n \times \partial_2 w_n) - \partial_2(w_n \times \partial_1 w_n)$,

$$\int_{\mathbb{T}\times D_1} (\partial_2 w_n, i\partial_1 w_n)\phi(x_2,x_3)\, dx = -\frac{1}{2}\int \partial_2(w_n \times \partial_1 w_n)\phi = \frac{1}{2}\int (iw_n, \partial_1 w_n)\partial_2\phi.$$

The case of the other term (with $\partial_3 w_n$) is similar. To conclude, note that $\mu_n^1 := n^2 \int_{\mathbb{T}} |\partial_1 w_n|^2 \, dx_1$ and $\mu_n^2 := n^2 \int_{\mathbb{T}} (iw_n, \partial_1 w_n) \, dx_1$, extended by 0 outside $\bar{D}_1$, are bounded in $L^1(\mathbb{R}^2)$. Indeed, for the first one, this follows from (80), and for the second one, we write first

$$\int_{\mathbb{T}\times D_{R_0/n}(\beta_n)} n^2 \big|(iw_n, \partial_1 w_n)\big| \le nC_L \Big(\int n^2 |\partial_1 w_n|^2\Big)^{1/2} \Big(\frac{K}{n^2}\Big)^{1/2} \le K,$$

by (80) and Cauchy-Schwarz, and then, since $\rho_n \ge 1/2$ outside $C_{R_0/n}(\beta_n)$ and writing for a real-valued map ψ_n, x_1-periodic, $w_n = \rho_n e^{i\psi_n + i\theta}$, we have

$$\begin{aligned}\int_{D_1\setminus D_{R_0/n}(\beta_n)} n^2 \Big|\int_{\mathbb{T}} (iw_n, \partial_1 w_n) \, dx_1\Big| &= \int_{D_1\setminus D_{R_0/n}(\beta_n)} n^2 \Big|\int_{\mathbb{T}} \rho_n^2 \partial_1 \psi_n \, dx_1\Big| \\ &= \varepsilon \int_{D_1\setminus D_{R_0/n}(\beta_n)} \Big|\int_{\mathbb{T}} \frac{\rho_n^2 - 1}{\delta} (n\partial_1\psi_n) \, dx_1\Big| \le K \int_{\mathbb{T}\times D_1} \frac{(\rho_n^2-1)^2}{\delta^2} + n^2|\partial_1 w_n|^2 \le K.\end{aligned}$$

Therefore, we may assume that μ_n^1 and μ_n^2 weakly converge as measures to μ_*^1 and μ_*^2 respectively. From the strong convergence result of Step 3 (as in Step 4), we deduce that the supports of μ_*^1 and μ_*^2 is in fact included in $\{\beta_*\}$. As a consequence, there exists $\lambda = \lambda(\varepsilon, L) \in \mathbb{R}$ such that (104) is satisfied for any $\phi \in \mathcal{C}^1(\bar{D}_1)$. The convergence in the distributional sense for Λ_n then follows. Concerning the L^p bound, we write by (104)

$$\Lambda_n = \frac{\partial}{\partial z}\Big(\frac{1}{2\pi}\log|(x_2,x_3)| * (\mu_n^2 - \mu_n^1)\Big),$$

and since $\mu_n^1 - \mu_n^2$ is bounded in $L^1(\mathbb{R}^2)$ and with compact support in $\bar{D}_1$, we deduce that

$$\frac{1}{2\pi}\log|(x_2,x_3)| * (\mu_n^2 - \mu_n^1) \qquad \text{is bounded in } W^{1,p}_{loc}(\mathbb{R}^2), \text{ for } 1 \le p < 2,$$

and thus Λ_n is bounded $L^p(D_1)$ for $1 \le p < 2$. The proof of Step 5 is complete. □

Step 6: We prove that $\boldsymbol{\kappa = 0}$ and $\boldsymbol{\beta_* = 0}$. We follow chapter VII of [BBH2] (the proofs of Theorem VII.1, Step 1, and Theorem VII.2). We introduce the Hopf differential defined in $\mathbb{T}\times D_1$

$$\omega_n := \big|\partial_2 w_n\big|^2 - \big|\partial_3 w_n\big|^2 - 2i(\partial_2 w_n, \partial_3 w_n),$$

where, we recall, $(.,.)$ is the scalar product in $\mathbb{R}^2 \simeq \mathbb{C}$. A straightforward computation shows that since w_n satisfies (82), then

$$\frac{\partial \omega_n}{\partial \bar{z}} = \frac{\partial}{\partial z}(2W_n) + 2(\partial_2 w_n, \partial_1 \Upsilon_n) - 2i(\partial_3 w_n, \partial_1\Upsilon_n) = \frac{\partial}{\partial z}(2W_n) + 2q_n(x), \tag{105}$$

where $2\frac{\partial}{\partial z} = (\partial_2 + i\partial_3)$. Identity (105) has to be compared with (5) in Step 1 of [BBH2]. We define also $\overline{W}_n = W_n$ in $\mathbb{T}\times D_1$ extended by 0 outside $\mathbb{T}\times D_1$ in $\mathbb{T}\times\mathbb{R}^2$, and define the distribution $T := \frac{\partial}{\partial z}(\frac{1}{\pi z})$. We consider $\alpha_n := T * \int_{\mathbb{T}} \overline{W}_n \, dx_1$ in the sense of distributions. Furthermore, by definition of Λ_n, we have

$$\int_{\mathbb{T}} q_n \, dx_1 = -\Delta\Lambda_n = -4\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z}\Lambda_n.$$

Therefore, by (105), we have in D_1,

$$\frac{\partial}{\partial\bar{z}}\Big(\int_{\mathbb{T}} \omega_n \, dx_1 - 2\alpha_n\Big) = 2\int_{\mathbb{T}} q_n(x) \, dx_1 = -8\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z}\Lambda_n. \tag{106}$$

Let us denote $f_n := \int_{\mathbb{T}} \omega_n \, dx_1 - 2\alpha_n + 8\frac{\partial}{\partial z}\Lambda_n$. By Step 5, $\frac{\partial}{\partial z}\Lambda_n$ is bounded in $L^p(D_1)$, $1 \le p < 2$. From Step 3, $\int_{\mathbb{T}} \omega_n \, dx_1$ is bounded in $L^\infty_{loc}(D_1 \setminus \{\beta_*\})$, thus in $L^p_{loc}(D_1\setminus\{\beta_*\})$, $1 \le p < 2$. Moreover, as for the claim in [BBH2], α_n is bounded in $L^\infty_{loc}(D_1 \setminus \{\beta_*\})$. Consequently, f_n is, by (106), a holomorphic function in D_1 bounded in $L^p_{loc}(D_1 \setminus \{\beta_*\})$, $1 \le p < 2$ thus bounded in $\mathcal{C}^k_{loc}(D_1)$ for any $k \in \mathbb{N}$ (by the Cauchy formula and an averaging argument), and we may then assume, up to another subsequence, that

$$f_n \to f_* \quad \text{ in } \mathcal{C}^k_{loc}(D_1) \;\; \forall k \in \mathbb{N}. \tag{107}$$

Since, by Step 3, $\int_{\mathbb{T}} \overline{W}_n \, dx_1$ converges as measure (up to a subsequence) to $m\delta_{\{\beta_*\}}$, we have

$$\alpha_n \to \alpha_* = mT * \delta_{\{\beta_*\}} = -\frac{m}{\pi(z-\beta_*)^2} \qquad \text{in } \mathcal{D}'(D_1). \tag{108}$$

Finally, combining Step 5, (107) and (108), we have in $\mathcal{D}'(D_1)$

$$\int_{\mathbb{T}} \omega_n \, dx_1 = f_n + 2\alpha_n - 8\frac{\partial \Lambda_n}{\partial z} \to \omega_* := f_* + 2\alpha_* - 8\frac{\partial \Lambda_*}{\partial z}, \tag{109}$$

and in view of Step 3, this convergence holds in $\mathcal{C}^k_{loc}(D_1 \setminus \{\beta_*\})$, $\forall k \in \mathbb{N}$ and ω_* is 2π times the Hopf differential of $\tilde{w}_*$ in $D_1 \setminus \{\beta_*\}$. To conclude, note that, by Step 5,

$$\Lambda_* = \frac{\lambda}{2\pi} \frac{\partial}{\partial \bar{z}} \log|(x_2, x_3)| * \delta_{\beta_*},$$

thus

$$\frac{\partial \Lambda_*}{\partial z} = \frac{\lambda}{2\pi} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log|(x_2, x_3)| * \delta_{\beta_*} = \frac{\lambda}{8\pi} (\Delta \log|(x_2, x_3)|) * \delta_{\beta_*} = \frac{\lambda}{4} \delta_0 * \delta_{\beta_*} = \frac{\lambda}{4} \delta_{\beta_*}.$$

From (109), we then infer, in $\mathcal{D}'(D_1 \setminus \{\beta_*\})$, with $z = x_2 + ix_3$,

$$\omega_* = f_* + 2\alpha_* = f_* - \frac{2m}{\pi(z-\beta_*)^2}. \tag{110}$$

This has to be compared with (13) and (14) in [BBH2], chapter VII. From (98) and the fact that the canonical harmonic map w_0 writes $\frac{z-\beta_*}{|z-\beta_*|} e^{i\chi_1}$ for some harmonic map χ_1 in a neighborhood of β_*, we infer that

$$\tilde{w}_* = \frac{z-\beta_*}{|z-\beta_*|} \exp(i\kappa \log|z-\beta_*| + i\chi'),$$

for some smooth real harmonic map χ' near β_*. Computing then the Hopf differential of $\tilde{w}_*$ and comparing with (110), we obtain as in [BBH2] that for z near β_* and $z \neq \beta_*$

$$f_* - \frac{2m}{\pi(z-\beta_*)^2} = \omega_* = 2\pi \Big(\frac{\kappa - i}{z - \beta_*} + 2\frac{\partial \chi'}{\partial z} \Big)^2 = 2\pi \Big[\frac{(\kappa-i)^2}{(z-\beta_*)^2} + 4\frac{\kappa - i}{z-\beta_*} \frac{\partial \chi'}{\partial z} + 4\Big(\frac{\partial \chi'}{\partial z}\Big)^2 \Big].$$

Since f_* and χ' are continuous in a neighborhood of β_* (including β_*), multiplying by $(z-\beta_*)^2$ and letting $z \to \beta_*$, we obtain

$$2\pi(\kappa - i)^2 = -\frac{2m}{\pi}, \tag{111}$$

and then, multiplying by $z - \beta_*$ and letting $z \to \beta_*$, we deduce

$$8\pi(\kappa - i)\frac{\partial \chi'}{\partial z}(\beta_*) = 0. \tag{112}$$

From (111), we infer that $\kappa = 0$ and $m = \pi^2$, since $\kappa \in \mathbb{R}$, thus $\tilde{w}_* = w_0$ the canonical harmonic map. From (112), it follows $\frac{\partial \chi'}{\partial z}(\beta_*) = 0$, which means $\nabla \chi'(\beta_*) = 0$ since χ' is real-valued. This last condition is equivalent to the fact that β_* is a critical point to the renormalized energy (see [BBH2], chapter VIII). From Theorem VIII.6 in [BBH2], we then know that the only critical point of the renormalized energy with one vortex and the boundary map $e^{i\theta}$ is 0. Therefore, $\beta_* = 0$ and the proof is complete. □

Remark 3.2. With a little more work, one can show that $\lambda = 0$.

3.6 Proof of Theorem 2 completed

In order to complete the proof of Theorem 2, we notice that $U_\varepsilon \in Y_\varepsilon$ is already proved. Hence, we are just left with proving that U_ε is a "local minimizer", that is, in view of the scaling, that u_ε is also one. Therefore, we assume that there exist $R > 0$ and $v \in H^1_{loc}(\mathbb{T} \times \mathbb{R}^2, \mathbb{C})$ such that

$$v = u_\varepsilon \quad \text{outside } C_R, \quad p(v) = 2\pi^2 L^2 = p(u_\varepsilon) \quad \text{and} \quad E_\varepsilon(v, C_R) < E_\varepsilon(u_\varepsilon, C_R).$$

We recall that since $v = u_\varepsilon$ outside $\mathbb{T} \times D_R$, $p(v)$ is well-defined. Taking R larger if necessary, we may assume

$$|v| = |u_\varepsilon| \geq \frac{1}{2} \text{ outside } C_R.$$

If we had $v = e^{i\theta + i\varphi_0}$ outside $\mathbb{T} \times D_R$ where φ_0 is a real constant, for $n \geq R$, the restriction of $e^{-i\varphi_0}v$ to Ω_n would be a map in X_n having momentum $p(v) = 2\pi^2 L^2$ (since in that case, $(iv, \partial_1 v) = 0$ outside C_R) and energy strictly less than the one of the minimizer $u_{\varepsilon,n}$, which is a contradiction. For the general case, as in [BOS], we construct such a map.

Outside C_R, since $v = u_\varepsilon$, we may write

$$v = u_\varepsilon = \rho \exp(i\varphi + i\theta).$$

We then define the functions (using cylindrical coordinates),

$$\sigma(x) := \frac{2R - r}{R}, \qquad \tau(x) := \frac{3R - r}{R} \qquad \text{and} \qquad \mu_R := \frac{1}{|\{2R \leq r \leq 3R\}|} \int_{2R \leq r \leq 3R} \varphi$$

and then

$$\rho_R(x) := \sigma(x)\rho(x) + (1 - \sigma(x)), \qquad \varphi_R(x) := \tau(x)\varphi(x) + (1 - \tau(x))\mu_R.$$

We then set

$$v_R(x) := \begin{cases} v(x) & \text{if } r \leq R, \\ \rho_R(x) \exp(i\varphi(x) + i\theta) & \text{if } R \leq r \leq 2R, \\ \exp(i\varphi_R(x) + i\theta) & \text{if } 2R \leq r \leq 3R, \\ \exp(i\mu_R + i\theta) & \text{if } r \geq 3R. \end{cases}$$

We claim that, for a constant C independent of ε, $n \geq 3R$ and R

$$|p(v_R) - p(u_\varepsilon)| \leq C\varepsilon \int_{r \geq R} |\partial_1 u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \tag{113}$$

$$\text{and} \qquad \left| E_\varepsilon(u_\varepsilon, \Omega_n \setminus C_R) - E_\varepsilon(v_R, \Omega_n \setminus C_R) \right| \leq \frac{C_L}{R^\lambda} + C_L \sigma_n(\varepsilon). \tag{114}$$

Proof of the claim. We first note that, in the definition of p given in (77), we may let χ tend to the characteristic function of C_R (for $R \geq R_0$) to obtain

$$p(u_\varepsilon) = \int_{C_R} (iu, \partial_1 u) + \int_{\mathbb{T} \times \mathbb{R}^2 \setminus C_R} (\rho^2 - 1)\partial_1 \varphi.$$

Therefore,

$$|p(v_R) - p(u_\varepsilon)| \leq \left| \int_{R \leq r \leq 2R} (\rho - \rho_R)\partial_1 \varphi \right| + \left| \int_{2R \leq r \leq 3R} (\tau - \rho^2)\partial_1 \varphi \right| + \left| \int_{r \geq 3R} (\rho^2 - 1)\partial_1 \varphi \right|. \tag{115}$$

For the first term, by Cauchy-Schwarz, since $\rho \geq 1/2$,

$$\Big|\int_{R\leq r\leq 2R}(\rho-\rho_R)\partial_1\varphi\Big| \leq \int_{R\leq r\leq 2R}|1-\sigma|\cdot|1-\rho|\cdot|\partial_1\varphi| \leq \frac{\varepsilon}{2}\int_{R\leq r\leq 2R}\frac{(1-\rho^2)^2}{\varepsilon^2}+|\partial_1\varphi|^2$$
$$\leq C\varepsilon\int_{R\leq r\leq 2R}|\partial_1 u|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{116}$$

For the second term, note that since φ is periodic in the variable x_1 and $\partial_1\tau = 0$,

$$\int_{2R\leq r\leq 3R}(\tau-\rho^2)\partial_1\varphi = -\int_{2R\leq r\leq 3R}\rho^2\partial_1\varphi = \int_{2R\leq r\leq 3R}(1-\rho^2)\partial_1\varphi,$$

thus

$$\Big|\int_{2R\leq r\leq 3R}(\tau-\rho^2)\partial_1\varphi\Big| \leq C\varepsilon\int_{2R\leq r\leq 3R}|\partial_1 u|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{117}$$

Concerning the last term, write also

$$\Big|\int_{r\geq 3R}(\rho^2-1)\partial_1\varphi\Big| \leq C\varepsilon\int_{r\geq 3R}|\partial_1 u|^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{118}$$

Inserting (116), (117) and (118) into (115) yields (113).

Concerning the energy, we have similarly

$$|E_\varepsilon(u_\varepsilon,\Omega_n\backslash C_R)-E_\varepsilon(v_R,\Omega_n\setminus C_R)| \leq \int_{R\leq r\leq 2R}|\rho^2-\rho_R^2|\cdot\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2+\Big|\frac{(1-\rho^2)^2}{2\varepsilon^2}-\frac{(1-\rho_R^2)^2}{2\varepsilon^2}\Big|$$
$$+\Big|\int_{2R\leq r\leq 3R}\rho^2\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2-\big|\nabla\varphi_R+\frac{\vec{e}_\theta}{r}\big|^2\Big|+\Big|\int_{3R\leq r\leq n}\big|\frac{\vec{e}_\theta}{r}\big|^2-e_\varepsilon(u_\varepsilon)\Big|, \tag{119}$$

and we estimate each term in (119). First, notice that $|1-\rho_R| = |(1-\sigma_R)(1-\rho)| \leq |1-\rho|$, so

$$(1-\rho_R^2)^2 \leq C_L(1-\rho_R)^2 \leq C_L(1-\rho)^2 \leq C_L(1-\rho^2)^2,$$

and then, by the decay result (12) in Proposition 1,

$$\int_{R\leq r\leq 2R}\Big|\frac{(1-\rho^2)^2}{2\varepsilon^2}-\frac{(1-\rho_R^2)^2}{2\varepsilon^2}\Big| \leq C_L\int_{R\leq r\leq 2R}\frac{(1-\rho^2)^2}{2\varepsilon^2} \leq \frac{C_L}{R^\lambda}+C_L\sigma_n(\varepsilon).$$

Next, since $\big|\nabla\varphi+r^{-1}\vec{e}_\theta\big| \leq C_L$ and using Proposition 1 once again

$$\int_{R\leq r\leq 2R}|\rho^2-\rho_R^2|\cdot\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2 \leq C_L\varepsilon\int_{R\leq r\leq 2R}\frac{|\rho^2-1|}{\varepsilon}(|\nabla\varphi|+\frac{1}{r^2}) \leq C_L\int_{R\leq r\leq 2R}f_\varepsilon \leq \frac{C_L}{R^\lambda}+C_L\sigma_n(\varepsilon),$$

since $r^{-2}\in L^2(\{r\geq 1\})$. We then infer the estimate for the first term in (119)

$$\int_{R\leq r\leq 2R}|\rho^2-\rho_R^2|\cdot\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2+\Big|\frac{(1-\rho^2)^2}{2\varepsilon^2}-\frac{(1-\rho_R^2)^2}{2\varepsilon^2}\Big| \leq \frac{C_L}{R^\lambda}+C_L\sigma_n(\varepsilon). \tag{120}$$

For the second term, since $\partial_\theta\varphi_R = \partial_\theta\varphi$, expansion yields

$$\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2-\big|\nabla\varphi_R+\frac{\vec{e}_\theta}{r}\big|^2 = |\nabla\varphi|^2-|\nabla\varphi_R|^2,$$

thus,

$$\Big|\int_{2R\le r\le 3R}\rho^2\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2-\big|\nabla\varphi_R+\frac{\vec{e}_\theta}{r}\big|^2\Big|\le\Big|\int_{2R\le r\le 3R}(\rho^2-1)\cdot(|\nabla\varphi|^2-|\nabla\varphi_R|^2)\Big| \\ +\frac{1}{R}\int_{2R\le r\le 3R}|\rho^2-1|\cdot|\nabla\varphi|+\int_{2R\le r\le 3R}\frac{|\rho^2-1|}{r^2}. \tag{121}$$

In (121), we estimate the second term by Cauchy-Schwarz, with the decay result (12), and the third one by Cauchy-Schwarz also, since $r^{-2}\in L^2(\{r\ge R_0\})$, to obtain

$$\frac{1}{R}\int_{2R\le r\le 3R}|\rho^2-1|\cdot|\nabla\varphi|+\int_{2R\le r\le 3R}\frac{|\rho^2-1|}{r^2}\le\frac{C_L}{R^\lambda}+C_L\sigma_n(\varepsilon). \tag{122}$$

For the first term in (121), using $|\nabla\varphi_R|+|\nabla\varphi|\le C_L$, $\nabla\varphi-\nabla\varphi_R=(\tau-1)\nabla\varphi+(\varphi-\mu_R)\nabla\tau$ and $|\nabla\tau|=R^{-1}$, then Poincaré-Wirtinger inequality and finally the decay result (12), we deduce

$$\begin{aligned}\Big|\int_{2R\le r\le 3R}(\rho^2-1)\cdot(|\nabla\varphi|^2-|\nabla\varphi_R|^2)\Big| &\le\Big|\int_{2R\le r\le 3R}(\rho^2-1)\cdot(\nabla\varphi+\nabla\varphi_R,\nabla\varphi-\nabla\varphi_R)\Big|\\ &\le C_L\int_{2R\le r\le 3R}|\rho^2-1|\Big[|\tau-1|\cdot|\nabla\varphi|+\frac{|\varphi-\mu_R|}{R}\Big]\\ &\le C_L\varepsilon\int_{2R\le r\le 3R}|\nabla\varphi|^2+\frac{(\rho^2-1)^2}{\varepsilon^2}\\ &\le\frac{C_L}{R^\lambda}+C_L\sigma_n(\varepsilon).\end{aligned} \tag{123}$$

Inserting (122) and (123) into (121) yields

$$\Big|\int_{2R\le r\le 3R}\rho^2\big|\nabla\varphi+\frac{\vec{e}_\theta}{r}\big|^2-\big|\nabla\varphi_R+\frac{\vec{e}_\theta}{r}\big|^2\Big|\le\frac{C_L}{R^\lambda}+C_L\sigma_n(\varepsilon). \tag{124}$$

From Proposition 1, we know that the last term verifies

$$\Big|\int_{3R\le r\le n}\big|\frac{\vec{e}_\theta}{r}\big|^2-e_\varepsilon(u_\varepsilon)\Big|=\Big|\int_{3R\le r\le n}e_\varepsilon(u_\varepsilon)-2\pi^2\log\Big(\frac{n}{3R}\Big)\Big|\le\frac{C_L}{R^\lambda}+\sigma_n(\varepsilon). \tag{125}$$

Inserting (120), (124) and (125) into (119) yields (114) and concludes the proof of the claim. □

Hence, if $R\to+\infty$, we have $v_R=e^{i\mu_R}$ outside $\mathbb{T}\times D_{3R}$ and

$$p(v_R)\to p(v)=2\pi^2L^2.$$

We may then define for R sufficiently large

$$\hat{v}_R(x):=v_R(x_1,\lambda_R x_2,\lambda_R x_3),$$

where $\lambda_R\to 1$ is uniquely defined by the equality

$$p(\hat{v}_R)=2\pi^2L^2.$$

Furthermore, we recall that $v=u_\varepsilon$ for $r\ge R$, and in view of the claim (114), we have

$$|E_\varepsilon(u_\varepsilon,\Omega_n\setminus C_R)-E_\varepsilon(\hat{v}_R,\Omega_n\setminus C_R)|\le\frac{C_L}{R^\lambda}+C_L\sigma_n(\varepsilon)$$

thus for fixed (but large) R, we have for n sufficiently large

$$E_\varepsilon(\hat{v}_R,\Omega_n)<E_\varepsilon(u_\varepsilon,\Omega_n),$$

with $\hat{v}_R=e^{i\theta+i\mu_R}$ on $\partial\Omega_n$, where μ_R is a constant. We are led to the desired contradiction. □

4 Proof of Proposition 2

The proof of the existence of a minimizer is standard and relies on the weak lower semicontinuity of the energy E_ε on X_n and on the fact that the momentum is, by Rellich compactness, weakly sequentially continuous on H^1, that is if $u_k \rightharpoonup u$ weakly in H^1 as $k \to +\infty$, then $u_k \to u$ strongly in L^2 by compactness of $\bar{\Omega}_n$, hence

$$2\pi^2 L^2 = p(u_k) = \int_{\mathbb{T}\times D_n} (iu_k, \partial_1 u_k) \to \int_{\mathbb{T}\times D_n} (iu, \partial_1 u) = p(u).$$

The Lagrange multiplier is written $\frac{c_{\varepsilon,n}}{2}|\log\varepsilon| \in \mathbb{R}$ and we expect the speed $c_{\varepsilon,n}$ to be bounded. We give the proof of Proposition 2, providing a bound for the energy of $u_{\varepsilon,n}$ and a bound in $|\log\varepsilon|$ for

$$\int_{\Omega_n} |\partial_1 u_{\varepsilon,n}|^2 + \big|\nabla_{2,3}|u_{\varepsilon,n}|\big|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}.$$

4.1 Definition of the comparison map

For the proof of the upper bound for I_ε^n, we have to construct a comparison map, behaving like an helicoidal vortex. To that aim, we first prove the following lemma, stating that the projection of the nearest point from $\mathbb{T}\times\mathbb{R}^2$ onto $\mathcal{H}_L$ is well-defined on the L-neighborhood of $\mathcal{H}_L$, but first, notice that the Frenet basis for $\vec{\mathcal{H}_L}$ at the point $(\alpha, L\cos\alpha, L\sin\alpha) \in \mathcal{H}_L$ is given by

$$\left\{\begin{array}{rcll} \vec{\tau}(\alpha) & = & \frac{1}{\sqrt{1+L^2}} & (1, -L\sin\alpha,\ L\cos\alpha), \\ \vec{\beta}(\alpha) & = & & (0,\ -\cos\alpha,\ -\sin\alpha), \\ \vec{\nu}(\alpha) & = & \frac{1}{\sqrt{1+L^2}} & (L,\quad \sin\alpha,\ -\cos\alpha). \end{array}\right.$$

We then define the following map (note that it is defined for $\alpha\in\mathbb{R}$ and not for $\alpha\in\mathbb{T}$)

$$\Phi_L : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}\times\mathbb{R}^2$$

$$(\alpha,u,v) \mapsto (\alpha, L\cos\alpha, L\sin\alpha) + u\vec{\beta}(\alpha) + v\vec{\nu}(\alpha).$$

Lemma 4.1. *The map Φ_L is injective on $\mathbb{R}\times\bar{D}_L$ and, if $L\le 1/2$, on $\mathbb{T}\times\bar{D}_{1/2}$; it induces a map, still denoted Φ_L, from $\mathbb{T}\times\mathbb{R}^2$ into $\mathbb{T}\times\mathbb{R}^2$. Moreover,*

$$\det Jac(\Phi_L) = \frac{1+L(L-u)}{\sqrt{1+L^2}}.$$

Proof of Lemma 4.1. In view of the expression of the Frenet basis, for $(\alpha,u,v)\in\mathbb{R}\times\bar{D}_L$ and $x\in\mathbb{R}\times\mathbb{R}^2$, $\Phi_L(\alpha,u,v)=x$ if and only if

$$\left\{\begin{array}{rcl} \alpha + \frac{Lv}{\sqrt{1+L^2}} & = & x_1, \\ L\cos\alpha - u\cos\alpha + \frac{v}{\sqrt{1+L^2}}\sin\alpha & = & x_2, \\ L\sin\alpha - u\sin\alpha - \frac{v}{\sqrt{1+L^2}}\cos\alpha & = & x_3. \end{array}\right. \tag{126}$$

For $(u,v)\in\bar{D}_L$, we have $L-u\ge 0$, thus we may write

$$\Big(L-u, \frac{v}{\sqrt{1+L^2}}\Big) = \rho(\cos\varphi, \sin\varphi) \tag{127}$$

for a $\rho\ge 0$ and a phase $\varphi\in[-\frac{\pi}{2},\frac{\pi}{2}]$, since $L-u\ge 0$, well-defined except for $(u,v)=(L,0)$. Using cylindrical coordinates (x_1,r,θ) with $\theta\in\mathbb{R}$ for x, the two last equations in (126) become

$$\left\{\begin{array}{rcccl} \rho\cos(\alpha-\varphi) & = & x_2 & = & r\cos\theta \\ \rho\sin(\alpha-\varphi) & = & x_3 & = & r\sin\theta, \end{array}\right. \tag{128}$$

which yields

$$r = \rho \qquad \text{and} \qquad \alpha - \varphi = \theta \bmod 2\pi. \tag{129}$$

Substituting (129) in the first equation in (126) yields

$$x_1 = \alpha + Lr\sin\varphi = \alpha + Lr\sin(\alpha - \theta). \tag{130}$$

We conclude noticing that, for fixed (r, θ), the map

$$\psi \mapsto \psi + Lr\sin(\psi - \theta) \tag{131}$$

is smoothly increasing on the set $\cup_{k\in\mathbb{Z}}[\theta - \frac{\pi}{2} + 2k\pi, \theta + \frac{\pi}{2} + 2k\pi]$ (since sin is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$), thus relations (129) and (130) define at most one couple $(k, \varphi) \in \mathbb{Z} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that, if $\alpha = \theta + \varphi + 2k\pi$, then

$$\alpha + Lr\sin\varphi = x_1,$$

which proves that Φ_L is injective and concludes the proof in the first case.

For the second case, we may also write $(L - u, \frac{v}{\sqrt{1+L^2}}) = \rho(\cos\varphi, \sin\varphi)$, but for $(u, v) \in \bar{D}_{1/2}$ now, we do not know that $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. However, we may use the fact that $Lr \leq \sqrt{5}/4 < 1$. Indeed, if $(u, v) \in \bar{D}_{1/2}$ and $L \leq 1/2$, we deduce from (127) and the equality $r = \rho$ that

$$r^2 = \rho^2 = (L - u)^2 + \frac{v^2}{1 + L^2} \leq 1 + \frac{1}{4} = \frac{5}{4},$$

thus $r \leq \frac{\sqrt{5}}{2}$, which implies $0 \leq Lr \leq \frac{\sqrt{5}}{4} < 1$. Therefore, the map (131) writes Identity plus a perturbation whose lipschitz constant is < 1. Hence, it is a smooth increasing diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.
For the computation of the jacobian, it suffices to write

$$Jac(\Phi_L) = \begin{pmatrix} 1 & 0 & \frac{L}{\sqrt{1+L^2}} \\ -(L-u)\sin\alpha + \frac{v}{\sqrt{1+L^2}}\cos\alpha & -\cos\alpha & \frac{\sin\alpha}{\sqrt{1+L^2}} \\ (L-u)\cos\alpha + \frac{v}{\sqrt{1+L^2}}\sin\alpha & -\sin\alpha & \frac{-\cos\alpha}{\sqrt{1+L^2}} \end{pmatrix},$$

and the computation of the determinant follows. □

From its definition, it is then clear that Φ_L is a diffeomorphism from $\mathbb{T} \times \bar{D}_L$ onto the closed L-neighborhood (or 1/2-neighborhood if $L \leq 1/2$) of $\mathcal{H}_L$ in $\mathbb{T} \times \mathbb{R}^2$. For x in this neighborhood, the closest point of x on $\mathcal{H}_L$ is the point $(\alpha, L\cos\alpha, L\sin\alpha) \in \mathcal{H}_L$ and $\|(u, v)\| = dist(x, \mathcal{H}_L)$. In particular, the projection of the nearest point onto $\mathcal{H}_L$ is always well-defined in the 1/2-neighborhood of $\mathcal{H}_L$.

We then come back to the definition of the comparison map. For $R > 0$ and $0 < \varepsilon \leq 1/4$, we define $w_{\varepsilon,R}$ in the 1/2-neighborhood of $\mathcal{H}_R$, denoted $\mathcal{H}_R^{1/2}$, by setting, with

$$\Phi_R^{-1}(x) = (\alpha, u, v),$$

$$w_{\varepsilon,R}(x) := \begin{cases} \varepsilon^{-1}(u + iv) & \text{if } \|(u, v)\| \leq \varepsilon, \\ \frac{u+iv}{|u+iv|} & \text{if } \varepsilon \leq \|(u, v)\| \leq 1/2, \end{cases}$$

which is the usual test function constructed with the projection on the orthogonal plane to the curve $\mathcal{H}_R$. The function $w_{\varepsilon,R}$ has therefore a degree one around $\vec{\mathcal{H}}_R$, and is of modulus one in $\mathcal{H}_R^{1/2} \setminus \mathcal{H}_R^{\varepsilon}$. We also define $w_{\varepsilon,R}$ outside $\mathbb{T} \times D_{L+2}$ by

$$w_{\varepsilon,R}(x) := e^{i\theta} \qquad \text{if} \quad L + 2 \leq r \leq n.$$

We are then just left with defining $w_{\varepsilon,R}$ on $\mathbb{T} \times D_{L+2} \setminus \mathcal{H}_R^{1/2}$, which is done in the following lemma.

Lemma 4.2. *For $0 \leq R \leq L+1$ and $0 < \varepsilon \leq 1/4$, the map $w_{\varepsilon,R}$, defined on $\mathbb{T} \times \partial D_{L+2} \cup \mathcal{H}_R^{1/2}$, admits an (helicoidally symmetric) extension to $\mathbb{T} \times D_{L+2} \setminus \mathcal{H}_R^{1/2}$, still denoted $w_{\varepsilon,R}$, $\mathbb{S}^1$-valued and having an energy $\leq C_L$ on $\mathbb{T} \times D_{L+2} \setminus \mathcal{H}_R^{1/2}$.*

Proof of Lemma 4.2. The proof of Lemma 4.2 follows the one of Theorem I.3 in chapter I of [BBH2], therefore, we only sketch the proof. The energy of the extension is related to the energy of the solution of the elliptic problem for the (closed) 2-form Ψ_ε, where h denotes the restriction of $w_{\varepsilon,R}$ to $\partial\mathcal{H}_R^{1/2}$ (i.e. $h = \frac{u+iv}{|u+iv|}$),

$$\begin{cases} -\Delta\Psi_\varepsilon &= 0 & \text{in } \mathbb{T} \times D_{L+2} \setminus \bar{\mathcal{H}}_R^{1/2}, \\ (d^*\Psi_\varepsilon)_\top &= h \times dh & \text{on } \partial\mathcal{H}_R^{1/2}, \\ (d^*\Psi_\varepsilon)_\top &= (d\theta)_\top & \text{on } \mathbb{T} \times \partial D_{L+2}, \\ (\Psi_\varepsilon)_\top &= 0 & \text{on } \mathbb{T} \times \partial D_{L+2} \cup \partial\mathcal{H}_R^{1/2}. \end{cases}$$

Since $(d\theta)_\top$ and $h \times dh$ are uniformly bounded, we infer $\|\nabla\Psi_\varepsilon\|_{L^2(\mathbb{T}\times D_{L+2}\setminus\bar{\mathcal{H}}_R^{1/2})} \leq C_L$. The conclusion then follows as in [BBH2], since the capacity of $\mathcal{H}_R^{1/2}$ in $\mathbb{T} \times D_{L+2}$ is $\leq C_L$ for $R \leq L+1$. $\square$

The following lemma summarizes the estimates concerning the energy and momentum of $w_{\varepsilon,R}$.

Lemma 4.3. *For $0 < \varepsilon \leq 1/4$, $n \in \mathbb{N}$ and $0 < \varepsilon \leq R \leq L+1 < L+2 \leq n$, the following inequalities hold for a constant C_L depending only on L and a function ρ such that $|\rho(s)| \leq C_L s$ for all $0 \leq s \leq 1$.*

(*i*) $\frac{1}{4\varepsilon^2}\int_{\Omega_n}(1-|w_{\varepsilon,R}|^2)^2 \leq C_L$,

(*ii*) $\frac{1}{2}\int_{\Omega_n}|\nabla w_{\varepsilon,R}|^2 \leq 2\pi^2\log n + 2\pi^2\sqrt{1+R^2}|\log\varepsilon| + C_L$,

(*iii*) $\int_{\Omega_n}(iw_{\varepsilon,R}, \partial_1 w_{\varepsilon,R}) = 2\pi^2R^2\Big(1+\rho(\frac{\varepsilon}{R})\Big)$.

To prove the upper bound (19), note that in view of (*iii*), since $L > 0$, there exists for ε sufficiently small ($\varepsilon < L$) $R = R(\varepsilon)$ such that

$$p(w_{\varepsilon,R}) = 2\pi^2L^2 \qquad \text{and} \qquad |R(\varepsilon) - L| \leq C_L\varepsilon.$$

Hence, this $w_{\varepsilon,R(\varepsilon)}$ satisfies, by (*i*) and (*ii*),

$$E_\varepsilon(w_{\varepsilon,R(\varepsilon)}) \leq 2\pi^2\log n + 2\pi^2\sqrt{1+R(\varepsilon)^2}|\log\varepsilon| + C_L + C_LR(\varepsilon) \leq 2\pi^2\log n + 2\pi^2\sqrt{1+L^2}|\log\varepsilon| + C_L,$$

which proves (19). We turn now to the proof of Lemma 4.3.

4.2 Estimates for the comparison map

Here, we prove the estimates of Lemma 4.3 for the energy and the momentum of the map $w_{\varepsilon,R}$.

Proof of (*i*) (the potential term). By construction, $w_{\varepsilon,R}$ is of modulus 1 outside $\mathcal{H}_R^\varepsilon$, and ≤ 1 in $\mathcal{H}_R^\varepsilon$ which is of measure $\leq C_L\varepsilon^2$. Therefore,

$$\int_{\Omega_n}\frac{(1-|w_{\varepsilon,R}|^2)^2}{4\varepsilon^2} = \int_{\mathcal{H}_R^\varepsilon}\frac{(1-|w_{\varepsilon,R}|^2)^2}{4\varepsilon^2} \leq C_L,$$

which is (*i*). $\square$

We will use during the proof the estimate

$$|\nabla w_{\varepsilon,R}| \leq \frac{C_L}{\varepsilon},$$

valid in $\mathcal{H}_R^{1/4}$. This estimate is due to the definition of $w_{\varepsilon,R}$ there, namely $w_{\varepsilon,R} = \varepsilon^{-1}(u+iv)$, and the fact that Φ_R is uniformly lipschitz for $0 \leq R \leq L+1$. Indeed, from the computations of Lemma 4.1, the first column of $Jac(\Phi_R)$ has a norm $1+(R-u)^2+v^2/(1+R^2) = 1+r^2 \leq C_L$, the two last columns have a norm 1 and (either $R-u \geq 0$ and $R \leq L+1$, either $R \leq \varepsilon \leq 1/4$ and $|R-u| \leq 1/2$)

$$\det Jac(\Phi_R) = \frac{1+R(R-u)}{\sqrt{1+R^2}} \geq C_L^{-1}.$$

Proof of (*ii*) (the gradient term). First, since $w_{\varepsilon,R} = e^{i\theta}$ if $r \geq L+2$, we have $|\nabla w_{\varepsilon,R}|^2 = r^{-2}$ for $r \geq L+2$, thus

$$\frac{1}{2}\int_{L+2\leq r\leq n} |\nabla w_{\varepsilon,R}|^2 = 2\pi^2 \int_{L+2}^{n} \frac{dr}{r} = 2\pi^2 \log\Big(\frac{n}{L+2}\Big) \leq 2\pi^2 \log n. \tag{132}$$

In order to estimate the gradient on $\mathcal{H}_R^\varepsilon$, we just write $|\nabla w_{\varepsilon,R}| \leq \frac{C_L}{\varepsilon}$, hence integrating on $\mathcal{H}_R^\varepsilon$ which is of measure $\leq C_L\varepsilon^2$, we have

$$\int_{\mathcal{H}_R^\varepsilon} |\nabla w_{\varepsilon,R}|^2 \leq C_L. \tag{133}$$

Furthermore, by definition, we have $|\nabla w_{\varepsilon,R}|^2 = \frac{1}{\|(u,v)\|^2}$ on $\mathcal{H}_R^{1/2}\backslash\mathcal{H}_R^\varepsilon$ thus, integrating, using the change of variables $x = \Phi_R(\alpha,u,v)$ (for which $Jac(\Phi_R) = \frac{1+R(R-u)}{\sqrt{1+R^2}} \geq 0$) and passing to polar coordinates $(u,v) \simeq (\rho,\varphi)$ yields

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{H}_R^{1/2}\backslash\mathcal{H}_R^\varepsilon} |\nabla w_{\varepsilon,R}|^2 &= \frac{\pi}{\sqrt{1+R^2}} \int_\varepsilon^{1/2}\int_0^{2\pi} (1+R(R-\rho\cos\varphi))d\varphi\frac{d\rho}{\rho} \\ &= 2\pi^2\sqrt{1+R^2}\int_\varepsilon^{1/2}\frac{d\rho}{\rho} \leq 2\pi^2\sqrt{1+R^2}|\log\varepsilon|.\end{aligned} \tag{134}$$

We conclude the proof of (i) combining Lemma 4.2, (132), (133) and (134). □

Proof of (*iii*) (the momentum). For the momentum, we integrate by parts, to obtain

$$p(w_{\varepsilon,R}) = \int_{\Omega_n} \langle Jw_{\varepsilon,R},\xi\rangle = \int_{\mathcal{H}_R^\varepsilon} \langle Jw_{\varepsilon,R},\xi\rangle,$$

since, outside $\mathcal{H}_R^\varepsilon$, $w_{\varepsilon,R}$ is lipschitz and of modulus 1 (if $1 \leq i < j \leq 3$, the two partial derivatives $\partial_{x_i}w_{\varepsilon,R}$ and $\partial_{x_j}w_{\varepsilon,R}$ are both tangent to $\mathbb{S}^1 \subset \mathbb{C}$ at the point $w_{\varepsilon,R} \in \mathbb{S}^1$ thus are colinear and then $Jw_{\varepsilon,R} = 0$). We then write

$$\xi = x_2dx_1\wedge dx_2 + x_3dx_1\wedge dx_3 = rdx_1\wedge dr,$$

so that $\langle Jw_{\varepsilon,R},\xi\rangle = r\partial_1 w_{\varepsilon,R}\times\partial_r w_{\varepsilon,R}$. From (127) and (129), we have

$$u = R - r\cos\varphi \qquad \text{and} \qquad v = \sqrt{1+R^2}r\sin\varphi, \tag{135}$$

which yields

$$\frac{\partial u}{\partial x_1} = r\sin\varphi\frac{\partial\varphi}{\partial x_1}, \qquad\qquad \frac{\partial v}{\partial x_1} = \sqrt{1+R^2}r\cos\varphi\frac{\partial\varphi}{\partial x_1}, \tag{136}$$

$$\frac{\partial u}{\partial r} = -\cos\varphi + r\sin\varphi\frac{\partial \varphi}{\partial r}, \qquad \frac{\partial v}{\partial r} = \sqrt{1+R^2}r\cos\varphi\frac{\partial\varphi}{\partial r} + \sqrt{1+R^2}\sin\varphi. \tag{137}$$

In view of (129), we have

$$\frac{\partial\varphi}{\partial x_1} = \frac{\partial\alpha}{\partial x_1} \quad \text{and} \quad \frac{\partial\varphi}{\partial r} = \frac{\partial\alpha}{\partial r}. \tag{138}$$

Moreover, from (130), we obtain

$$\frac{\partial\alpha}{\partial x_1} + Rr\cos\varphi\frac{\partial\varphi}{\partial x_1} = 1 \quad \text{and} \quad \frac{\partial\alpha}{\partial r} + R\sin\varphi + Rr\cos\varphi\frac{\partial\varphi}{\partial r} = 0. \tag{139}$$

Combining relations (136), (137), (138) and (139), we deduce, by (135) and recalling $R - u \geq 0$,

$$\frac{\partial\varphi}{\partial x_1} = (1 + R(R-u))^{-1} \quad \text{and} \quad \frac{\partial\varphi}{\partial r} = -(R\sin\varphi)(1+R(R-u))^{-1}$$

and therefore

$$\frac{\partial u}{\partial x_1} = v(1+R^2)^{-1/2}(1+R(R-u))^{-1}, \qquad \frac{\partial v}{\partial x_1} = \sqrt{1+R^2}(R-u)(1+R(R-u))^{-1},$$

$$r\frac{\partial u}{\partial r} = -(R-u) - \frac{Rv^2}{1+R^2}(1+R(R-u))^{-1}, \qquad r\frac{\partial v}{\partial r} = v(1+R(R-u))^{-1}.$$

We thus infer that, since $w_{\varepsilon,R} = \varepsilon^{-1}(u+iv)$ in $\mathcal{H}_R^\varepsilon$,

$$\langle Jw_{\varepsilon,R}, \xi\rangle = r\partial_1 w_{\varepsilon,R} \times \partial_r w_{\varepsilon,R} = \varepsilon^{-2}\Big(\frac{\partial u}{\partial x_1}r\frac{\partial v}{\partial r} - \frac{\partial v}{\partial x_1}r\frac{\partial u}{\partial r}\Big)$$

$$= \varepsilon^{-2}(1+R^2)^{-1/2}(1+R(R-u))^{-2}\Big[v^2 + (R-u)\Big(Rv^2 + (1+R^2)(R-u)(1+R(R-u))\Big)\Big].$$

Next, we integrate and successively use the change of variables $x = \Phi_R(\alpha, u, v)$ (we have computed its jacobian $\det Jac(\Phi_R) = (1+R^2)^{-1/2}(1+R(R-u))$ in Lemma 4.1) and use polar coordinates $(u,v) \simeq (\varepsilon\rho, \psi)$ to obtain, with $\delta := \varepsilon/R$,

$$p(w_{\varepsilon,R}) = \frac{2\pi}{1+R^2}\int_0^{2\pi}\int_0^1 (1+R^2(1-\rho\delta\cos\psi))^{-1}$$

$$\Big[R^2\delta^2\rho^2\sin^2\psi + R(1-\delta\rho\cos\psi)\Big(R^3\delta^2\rho^2\sin^2\psi + R(1+R^2)(1-\delta\rho\cos\psi)(1+R^2(1-\delta\rho\cos\psi))\Big)\Big]\rho\, d\rho d\psi.$$

To conclude the proof of *(iii)*, we notice that the integrand is a smooth function in the variables (δ, R, ρ, ψ) in $[0,1]\times[0,L+1]\times[0,1]\times[0,2\pi]$ since there $1 + R^2(1-\rho\delta\cos\psi) \geq 1$, and the integral has value for $\delta = 0$

$$\frac{2\pi}{1+R^2}\int_0^{2\pi}\int_0^1 (1+R^2)^{-1}(R^2(1+R^2)^2)\rho\, d\rho d\psi = 2\pi^2 R^2.$$

Hence there exists $\rho : [0,1] \to \mathbb{R}$, such that $|\rho(s)| \leq C_L s$ for $s \in [0,1]$, and for $0 < \varepsilon \leq R \leq L+1$

$$p(w_{\varepsilon,R}) = 2\pi^2 R^2\Big(1 + \rho(\frac{\varepsilon}{R})\Big)$$

which is *(iii)*. $\square$

4.3 A preliminary result

In this subsection, we present a preliminary result concerning a lower bound for the Ginzburg-Landau functional taking into account the degree at infinity. These lower bounds, as in [San2] (see also [J2]), will provide directly the desired result (compare with Theorem 3 in [San2]). Comparing with [J2], it has the advantage of separating the energies of the modulus and of the phase globally in Ω_n, which is crucial for our problem. We consider a lipschitz map

$$w : \mathbb{T} \times D_n \to \mathbb{C}$$

satisfying $w = g = e^{i\theta}$ on $\mathbb{T} \times \partial D_n$. We follow very closely the lines of [San2]. We will need to extend w on a larger domain. In view of the boundary condition, it is natural to extend w by setting

$$w := e^{i\theta} \qquad \text{in } \mathbb{T} \times (D_{3n} \setminus D_n).$$

The energy of the new w is then the old one plus $\pi \log \frac{3n}{n} = \pi \log 3$. We recall the definition of the radius from [San2] (in our context). Let $K \subset \mathbb{R}^2$ be compact. We define the radius $|K|$ of K by

$$|K| := \inf \Big\{ \sum_{i=1}^{n} r_i,\ n \in \mathbb{N},\ a_i \in \mathbb{R}^2, K \subset \cup_{i=1}^{n} D(a_i, r_i) \Big\}.$$

We will make use of the following Proposition taken from [San2].

Proposition 4.1. *Assume $\Omega \subset \mathbb{R}^2$ is a bounded open set and $\omega \subset \Omega$ is compact at distance greater than $2\rho > 0$ from $\partial\Omega$. Then, for any $v : \Omega \setminus \omega \to \mathbb{S}^1 \subset \mathbb{C}$ having degree $d \in \mathbb{Z}$ on $\partial\Omega$,*

$$\frac{1}{2} \int_{\Omega \setminus \omega} |\nabla v|^2 \geq \pi |d| \log\Big(\frac{\rho}{|\omega|}\Big).$$

For the extended map w, we will have $\Omega = D_{3n}$, $\rho = n$ and $d = 1$, and we deduce the following corollary.

Corollary 4.1. *Let $\omega \subset \bar{D}_n$ be compact and $v : D_n \setminus \omega \to \mathbb{S}^1 \subset \mathbb{C}$ such that $v(z) = z/|z| = e^{i\theta}$ on ∂D_n, then*

$$\frac{1}{2} \int_{D_n \setminus \omega} |\nabla v|^2 \geq \pi \log\Big(\frac{n}{|\omega|}\Big) - \pi \log 3.$$

We deduce from Corollary 4.1 the main lower bound for a map having a degree one at infinity.

Lemma 4.4. *Let $H \subset \bar{D}_n$ be compact and $w : \mathbb{T} \times D_n \to \mathbb{C}$ be a lipschitz map such that $w = e^{i\theta}$ on $\mathbb{T} \times \partial D_n$. Then, there exists C, independent of $0 < \varepsilon \leq 1/2$, $n \in \mathbb{N}^*$ and H, such that*

$$\frac{1}{2} \int_{\mathbb{T} \times (D_n \setminus H)} |\nabla_{2,3} w|^2 + \frac{(1 - |w|^2)^2}{2\varepsilon^2} \geq 2\pi^2 \log n + 2\pi^2 (1 - t_*^2) |\log \varepsilon| - 2\pi^2 t_*^2 \log(|H|) - C,$$

with

$$t_* := \sqrt{1 + \Big(\frac{\pi\varepsilon}{2\sqrt{2}|H|}\Big)^2} - \frac{\pi\varepsilon}{2\sqrt{2}|H|} \in [0, 1]$$

and the convention that $t_* = t_*^2 \log(|H|) = 0$ if $|H| = 0$.

Notations: We will use the following notations. For $t \geq 0$ and $a \in \mathbb{T}$, set

$$\Omega_t^a := \{y \in D_n \setminus H,\ |w(a, y)| > t\}, \qquad \omega_t^a := \{y \in D_n \setminus H,\ |w(a, y)| \leq t\}, \qquad w^a := w(a, .),$$

$$\gamma_t^a := \partial\Omega_t^a \setminus \partial D_n = \partial\omega_t^a \qquad \text{and} \qquad E_\varepsilon^a(w) := \frac{1}{2}\int_{\{a\}\times(D_n\setminus H)} |\nabla_{2,3}w^a|^2 + \frac{(1-|w^a|^2)^2}{2\varepsilon^2}.$$

For $a \in \mathbb{T}$ and $t \geq 0$, consider the functions

$$\Theta^a(t) := \frac{1}{2}\int_{\Omega_t^a} \Big|\nabla\Big(\frac{w^a}{|w^a|}\Big)\Big|^2 dy \qquad \text{and} \qquad \nu^a(t) := \int_{\gamma_t^a} |\nabla|w^a||d\mathcal{H}^1.$$

Proof of Lemma 4.4. First, we fix $a \in \mathbb{T}$. Since w^a is lipschitz, the coarea formula gives

$$E_\varepsilon^a(w) = \frac{1}{2}\int_0^{+\infty} \Big[\int_{\gamma_t^a} |\nabla|w^a|| + \frac{(1-t^2)^2}{2\varepsilon^2|\nabla|w^a||}\, d\mathcal{H}^1\Big] - 2t^2(\Theta^a)'(t)\; dt.$$

By Cauchy-Schwarz inequality,

$$\int_{\gamma_t^a} \frac{1}{|\nabla|w^a||}\, d\mathcal{H}^1 \geq \frac{\mathcal{H}^1(\gamma_t^a)^2}{\nu^a(t)}$$

and from the definition of the radius,

$$\mathcal{H}^1(\gamma_t^a) \geq 2diam(\gamma_t^a) \geq 4|\omega_t^a|,$$

since if $u, v \in \gamma_t^a$ are such that $diam(\gamma_t^a) = |u-v|$, then $\omega_t^a \subset D((u+v)/2, |u-v|/2)$ and therefore $|\omega_t^a| \leq |u-v|/2 = diam(\gamma_t^a)/2$. It follows from the inequality $(a^2+b^2)/2 \geq ab$ that

$$\begin{aligned} E_\varepsilon^a(w) &\geq \frac{1}{2}\int_0^{+\infty} \nu^a(t) + \frac{8(1-t^2)^2|\omega_t^a|^2}{\varepsilon^2\nu^a(t)}\; dt - \int_0^{+\infty} t^2(\Theta^a)'(t)\; dt \\ &\geq \int_0^{+\infty} \frac{2\sqrt{2}}{\varepsilon}|1-t^2|\cdot|\omega_t^a|\; dt - \int_0^{+\infty} t^2(\Theta^a)'(t)\; dt. \end{aligned}$$

We integrate by parts the last term. Since w is lipschitz, Θ^a has compact support in $\mathbb{R}_+$ and is locally lipschitz on $\mathbb{R}_+^*$ (note that $\Theta^a(0) = +\infty$). Since $\Theta^a \geq 0$ and $-(\Theta^a)' \geq 0$, we have by monotone convergence

$$\begin{aligned} -\int_0^{+\infty} t^2(\Theta^a)'(t)\; dt = \lim_{\eta\to 0} -\int_\eta^{+\infty} t^2(\Theta^a)'(t)\; dt &= \lim_{\eta\to 0}\Big(2\int_\eta^{+\infty} t\Theta^a(t)\; dt + \eta^2\Theta^a(\eta)\Big) \\ \geq \lim_{\eta\to 0} 2\int_\eta^1 t\Theta^a(t)\; dt &= 2\int_0^1 t\Theta^a(t)\; dt. \end{aligned}$$

From Corollary 4.1, we know that

$$\Theta^a(t) \geq -\pi\log\Big(\frac{|\omega_t^a \cup H|}{n}\Big) - \pi\log 3, \tag{140}$$

hence, since $|\omega_t^a \cup H| \leq |\omega_t^a| + |H|$,

$$E_\varepsilon^a(w) \geq \int_0^1 \frac{2\sqrt{2}}{\varepsilon}(1-t^2)|\omega_t^a| - 2t\pi\log\Big(\frac{|\omega_t^a|+|H|}{n}\Big)\; dt - C.$$

Next, we notice that, for fixed $\varepsilon > 0$, $t \in (0,1)$, the function $f(r) := 2\sqrt{2}\varepsilon^{-1}(1-t^2)r - 2t\pi\log\Big(\frac{r+|H|}{n}\Big)$, defined for $r > -|H|$ has a minimum for $r = r_* := 2^{-1/2}\pi\varepsilon t(1-t^2)^{-1} - |H|$ (note that $r_* > -|H|$), but it can occur that $r_* < 0$. If $r_* \geq 0$, then

$$\frac{2\sqrt{2}}{\varepsilon}(1-t^2)|\omega_t^a| - 2t\pi\log\Big(\frac{|\omega_t^a|+|H|}{n}\Big) = f(|\omega_t^a|) \geq f(r_*) \geq -2t\pi\log\Big(\frac{\pi t\varepsilon}{n\sqrt{2}(1-t^2)}\Big),$$

and if $r_* < 0$, f is increasing on $\mathbb{R}_+$ and then

$$\frac{2\sqrt{2}}{\varepsilon}(1-t^2)|\omega_t^a| - 2t\pi\log\Big(\frac{|\omega_t^a|+|H|}{n}\Big) = f(|\omega_t^a|) \geq f(0) = -2t\pi\log\Big(\frac{|H|}{n}\Big).$$

Moreover, we have $r_* \geq 0$ if and only if $t \geq t_*$, thus

$$\begin{aligned} E_\varepsilon^a(w) &\geq \Big(\int_0^{t_*} + \int_{t_*}^1\Big) \frac{2\sqrt{2}}{\varepsilon}(1-t^2)|\omega_t^a| - 2t\pi\log\Big(\frac{|\omega_t^a|+|H|}{n}\Big)\ dt - C \\ &\geq \int_0^{t_*} -2t\pi\log\Big(\frac{|H|}{n}\Big)\ dt - \int_{t_*}^1 2t\pi\log\Big(\frac{\pi t\varepsilon}{n\sqrt{2}(1-t^2)}\Big)\ dt - C \\ &\geq \pi t_*^2\log\Big(\frac{n}{|H|}\Big) + \pi(1-t_*^2)\log\Big(\frac{n}{\varepsilon}\Big) - C \end{aligned}$$

since $t\mapsto t\log\Big(t(1-t^2)^{-1}\Big) \in L^1(0,1)$. The conclusion follows integrating in $a\in\mathbb{T}$. $\square$

4.4 Proof of Proposition 2 completed

We are now in position to complete the proof of Proposition 2. We are just left with the (important) inequality (20). We will follow closely the lines of the proof of Theorem 3 in [San2]. We also denote, for the lipschitz map $u = u_{\varepsilon,n} : \mathbb{T}\times D_n \to \mathbb{C}$ having boundary condition $u = e^{i\theta}$ on $\mathbb{T}\times\partial D_n$ and for $a\in\mathbb{T}$,

$$T^a := -\int_0^{+\infty} t^2(\Theta^a)'(t)\ dt, \qquad N^a := \frac{1}{2}\int_{D_n} \big|\nabla_{2,3}|u^a|\big|^2 + \frac{(1-|u^a|^2)^2}{2\varepsilon^2}\ dx_2dx_3,$$

$$\tilde{T}^a := \int_0^1 2t\Theta^a(t)\ dt, \qquad I^a := \int_0^1 \frac{2\sqrt{2}}{\varepsilon}|\omega_t^a|(1-t^2)\ dt \ \text{ and } \ J^a := \int_0^1 2t\pi\log\Big(\frac{n}{|\omega_t^a|}\Big)\ dt - C.$$

From the proof of Lemma 4.4 with $H=\emptyset$ (so $t_*^2\log(|H|)=0$ and $r_*\geq 0$), we know that for any $a\in\mathbb{T}$,

$$J^a \leq \tilde{T}^a \leq T^a, \quad I^a \leq N^a \ \text{ and } \int_{\mathbb{T}}(I^a+\tilde{T}^a)\ da \geq \int_{\mathbb{T}}(I^a+J^a)\ da \geq 2\pi^2\log\Big(\frac{n}{\varepsilon}\Big) - C. \tag{141}$$

Moreover, we have by the upper bound (19)

$$E_\varepsilon(u) = \int_{\mathbb{T}}(T^a+N^a)\ da + \frac{1}{2}\int_{\Omega_n}|\partial_1 u|^2 \leq 2\pi^2\log n + 2\pi^2\sqrt{1+L^2}|\log\varepsilon| + C_L. \tag{142}$$

Writing $1 = \int_0^1 2t\ dt$, we deduce from (141) that for $a\in\mathbb{T}$

$$T^a - \pi\log\Big(\frac{n}{\varepsilon}\Big) \geq J^a - \pi\log\Big(\frac{n}{\varepsilon}\Big)\int_0^1 2t\ dt = \int_0^1 2t\pi\log\Big(\frac{\varepsilon}{|\omega_t^a|}\Big)\ dt - C,$$

and since $t\mapsto\log(1-t^2)\in L^1(0,1)$,

$$T^a - \pi\log\Big(\frac{n}{\varepsilon}\Big) \geq -\pi\int_0^1 2t\log\Big(\frac{2\sqrt{2}}{\varepsilon}|\omega_t^a|(1-t^2)\Big)\ dt - C.$$

By Jensen inequality applied with the concave function log and the interval $[0,1]$ with measure $2t\ dt$ (hence the total mass of $[0,1]$ is 1),

$$\int_0^1 2t\log\Big(\frac{2\sqrt{2}}{\varepsilon}|\omega_t^a|(1-t^2)\Big)\ dt \leq \log\Big(\int_0^1 4t\frac{\sqrt{2}}{\varepsilon}|\omega_t^a|(1-t^2)\ dt\Big) \leq \log\Big(\int_0^1 \frac{2\sqrt{2}}{\varepsilon}|\omega_t^a|(1-t^2)\ dt\Big) + \log 2.$$

We therefore deduce

$$T^a - \pi \log\Big(\frac{n}{\varepsilon}\Big) \geq -\pi \log\Big(\int_0^1 \frac{2\sqrt{2}}{\varepsilon}|\omega_t^a|(1-t^2)\ dt\Big) - C = -\pi \log I^a - C.$$

Adding N^a, with (141), integrating for $a \in \mathbb{T}$ and using (142), we infer

$$\begin{aligned} 2\pi^2 \log n + 2\pi^2\sqrt{1+L^2}|\log\varepsilon| + C_L &\geq \frac{1}{2}\int_{\Omega_n} |\partial_1 u|^2 + \int_{\mathbb{T}} (T^a + N^a)\ da \\ &\geq \frac{1}{2}\int_{\Omega_n} |\partial_1 u|^2\ dx + 2\pi^2 \log\Big(\frac{n}{\varepsilon}\Big) + \int_{\mathbb{T}} (N^a - \pi\log N^a)\ da - C, \end{aligned}$$

and thus

$$\frac{1}{2}\int_{\Omega_n} |\partial_1 u|^2 + \int_{\mathbb{T}} (N^a - \pi\log N^a)\ da \leq C_L|\log\varepsilon| + C_L.$$

We then use Jensen inequality with log again but on $\mathbb{T}$ with measure $da/(2\pi)$ to obtain

$$\int_{\mathbb{T}} \log N^a\ da = 2\pi \int_{\mathbb{T}} \log N^a\ \frac{da}{2\pi} \leq 2\pi \log\Big(\int_{\mathbb{T}} N^a\ \frac{da}{2\pi}\Big) = 2\pi\log\Big(\int_{\mathbb{T}} N^a\ da\Big) + C,$$

which implies

$$\frac{1}{2}\int_{\Omega_n} |\partial_1 u|^2 + \int_{\mathbb{T}} N^a\ da - 2\pi^2\log\Big(\int_{\mathbb{T}} N^a\ da\Big) \leq C_L|\log\varepsilon| + C_L,$$

from which we easily deduce

$$\frac{1}{2}\int_{\Omega_n} |\partial_1 u|^2 + \int_{\mathbb{T}} N^a\ da \leq C_L|\log\varepsilon|. \tag{143}$$

Estimate (20) comes from (143) and (141). □

5 Proofs of Lemmas 2 and 3

5.1 Proof of Lemma 2

We recall that Lemma 2 states that the two expressions integrated in the momentum of $u_{\varepsilon,n}$ and $v_{\varepsilon,n}$ are close (nearly in $L^1(\Omega_n)$). From (19) and Lemma 2.1, we know that

$$E_\varepsilon(v_{\varepsilon,n}) + \int_{\Omega_n} \frac{|\tilde{u} - v_{\varepsilon,n}|^2}{2\varepsilon} \leq I_\varepsilon^n \leq 2\pi^2\log n + 2\pi^2\sqrt{1+L^2}|\log\varepsilon| + C_L. \tag{144}$$

Since $v_{\varepsilon,n}$ is lipschitz and has value $g = e^{i\theta}$ on $\mathbb{T}\times\partial D_n$ we may apply the arguments of subsection 4.4 to $v_{\varepsilon,n}$ and obtain first the following lemma.

Lemma 5.1. *The map $v_{\varepsilon,n}$ satisfies, ω_t^a being defined for $v_{\varepsilon,n}$,*

$$\int_{\Omega_n} |\partial_1 v_{\varepsilon,n}|^2 + \frac{1}{2\varepsilon^2}(1-|v_{\varepsilon,n}|^2)^2 + \frac{|v_{\varepsilon,n} - \tilde{u}|^2}{\varepsilon} \leq C_L|\log\varepsilon|, \tag{145}$$

$$\int_{\mathbb{T}}\int_0^1 \varepsilon^{-1}|\omega_t^a|(1-t^2)\ dtda \leq C_L|\log\varepsilon|. \tag{146}$$

The proof is the same as for Lemma 4.4, just replace (142) by (144). Estimate (146) is then deduced as (143) and will be used in the proof of Corollary 1. We can therefore prove Lemma 2.

Proof of Lemma 2. Let $B \subset D_n$ be a measurable set. In view of the periodicity in the x_1 variable, integration by parts yields

$$\int_{\mathbb{T}\times B}(iv_{\varepsilon,n},\partial_1 v_{\varepsilon,n}) - \int_{\mathbb{T}\times B}(i\tilde{u},\partial_1\tilde{u}) = \int_{\mathbb{T}\times B}(i(v_{\varepsilon,n}-\tilde{u}),\partial_1(v_{\varepsilon,n}+\tilde{u})).$$

Thus, by Cauchy-Schwarz and (145),

$$\begin{aligned}\Big|\int_{\mathbb{T}\times B}(iv_{\varepsilon,n},\partial_1 v_{\varepsilon,n}) - \int_{\mathbb{T}\times B}(i\tilde{u},\partial_1\tilde{u})\Big| &\leq \int_{\mathbb{T}\times B}|\tilde{u}-v_{\varepsilon,n}|(|\partial_1\tilde{u}|+|\partial_1 v_{\varepsilon,n}|)\\ &\leq \sqrt{2}\Big(\int_{\mathbb{T}\times B}|\tilde{u}-v_{\varepsilon,n}|^2\Big)^{\frac12}\Big(\int_{\mathbb{T}\times B}|\partial_1\tilde{u}|^2+|\partial_1 v_{\varepsilon,n}|^2\Big)^{\frac12}\\ &\leq C_L\sqrt{\varepsilon}|\log\varepsilon|. \end{aligned}\tag{147}$$

We estimate similarly, since $|\partial_1\tilde{u}| \leq |\partial_1 u_{\varepsilon,n}|$ and using (20),

$$\begin{aligned}\Big|\int_{\mathbb{T}\times B}(i\tilde{u},\partial_1\tilde{u}) - \int_{\mathbb{T}\times B}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big| &\leq \int_{\mathbb{T}\times B}|u_{\varepsilon,n}-\tilde{u}|(|\partial_1\tilde{u}|+|\partial_1 u_{\varepsilon,n}|)\\ &\leq 2\Big(\int_{\mathbb{T}\times B}|\tilde{u}-u_{\varepsilon,n}|^2\Big)^{\frac12}\Big(\int_{\mathbb{T}\times B}|\partial_1 u_{\varepsilon,n}|^2\Big)^{\frac12}\\ &= 2\Big(\int_{(\mathbb{T}\times B)\cap\{|u|>1\}}(1-|u|^2)^2\Big)^{\frac12}\Big(\int_{\mathbb{T}\times B}|\partial_1 u_{\varepsilon,n}|^2\Big)^{\frac12}\\ &\leq C_L\varepsilon|\log\varepsilon|. \end{aligned}\tag{148}$$

Combining estimates (147) and (148) yields the result. $\square$

5.2 Proof of Lemma 3 : rough localization of the singular set

In this subsection, we prove Lemma 3 concerning the rough location of the singular set of $v_{\varepsilon,n}$, defined by $S := \{|v_{\varepsilon,n}| \leq 1/2\}$. We will make use of the following trivial observation.

Lemma 5.2. *Let $(D_i)_{i\in I}$ be a finite collection of closed disks in $\mathbb{R}^2$ of radii r_i. Then, there exists a finite collection of pairwise disjoint closed disks $(\tilde{D}_j)_{j\in J}$ in $\mathbb{R}^2$ of radii $\tilde{r}_j$ such that*

$$\cup_{i\in I}D_i \subset \cup_{j\in J}\tilde{D}_j,$$

the sets $(\{i\in I,\ D_i\subset\tilde{D}_j\})_{j\in J}$ induce a partition of I,

$$\sum_{j\in J}\tilde{r}_j \leq \sum_{i\in I}r_i$$

and $\sharp J \leq \sharp I$ with strict inequality unless $(D_i)_{i\in I}$ is pairwise disjoint.

Proof of Lemma 5.2. If $D_i \cap D_j \neq \emptyset$ for $i,j\in I$, $i\neq j$, then we replace them by a disk $\tilde{D}$ of radius r such that $D_i\cup D_j \subset \tilde{D}$ and $r \leq r_i + r_j$, and then delete the disks $D_k \subset \tilde{D}$ ($k\in I$, $k\neq i,j$); we repeat this until the collection is pairwise disjoint, which occurs in a finite number of steps since I is finite. $\square$

Proof of Lemma 3. In order to locate the singular set $S := \{|v_{\varepsilon,n}|\leq 1/2\}$ of $v_{\varepsilon,n}$, we consider the covering of S by the balls $B(x, 5\varepsilon/(4C_0))$, $x\in S$ (where C_0 is the constant in Lemma 2.1). By the Vitali´s covering theorem, there exists an at most countable family $(a_i)_{i\in I}$ in S such that

$$S \subset \cup_{i\in I}B(a_i, 5\varepsilon/(4C_0))$$

and

$$B(a_i,\varepsilon/(4C_0))\cap B(a_j,\varepsilon/(4C_0))=\emptyset \qquad \text{if } i\neq j.$$

The question is then to determine a bound for $\sharp I$. To that aim, from (146) in Lemma 5.1

$$\int_{\mathbb{T}}\int_0^1 \varepsilon^{-1}|\omega_t^a|(1-t^2)\ dtda \leq C_L|\log\varepsilon|,$$

there exists, by the mean-value formula, $\tau_*\in[3/4,1]$ such that

$$\int_{\mathbb{T}}|\omega_{\tau_*}^a|\ da\leq C_L\varepsilon|\log\varepsilon|. \tag{149}$$

For each $i\in I$, we have $|v_{\varepsilon,n}(a_i)|\leq 1/2$, so, since $|\nabla v_{\varepsilon,n}|\leq C_0/\varepsilon$,

$$B(a_i,\varepsilon/(4C_0))\subset\{|v_{\varepsilon,n}|\leq 3/4\}.$$

Hence, if $|a-a_i^1|\leq\varepsilon/(4C_0)$ (where $a_i=(a_i^1,a_i^2,a_i^3)$ and $|.|$ denotes the distance in $\mathbb{R}/(2\pi\mathbb{Z})$),

$$D\Big((a_i^2,a_i^3),\sqrt{(\varepsilon/(4C_0))^2-|a-a_i^1|^2}\Big)\subset\{|v_{\varepsilon,n}(a,.)|\leq 3/4\}\subset\omega_{\tau_*}^a$$

and since the balls $B(a_i,\varepsilon/(4C_0))$ are pairwise disjoint, we deduce

$$|\omega_{\tau_*}^a|\geq\sum_{i\in I}\chi_{\{|a-a_i^1|\leq\varepsilon/(4C_0)\}}\sqrt{(\varepsilon/(4C_0))^2-|a-a_i^1|^2},$$

where χ stands for the characteristic function. Integrating for $a\in\mathbb{T}$ yields

$$\int_{\mathbb{T}}|\omega_{\tau_*}^a|\ da\geq\sum_{i\in I}\int_{\mathbb{T}}\chi_{\{|a-a_i^1|\leq\varepsilon/(4C_0)\}}\sqrt{(\varepsilon/(4C_0))^2-|a-a_i^1|^2}\ da.$$

By periodicity, all the integrals are equal and have value (for $\varepsilon/(4C_0)<\pi$)

$$\int_{-\varepsilon/(4C_0)}^{\varepsilon/(4C_0)}\sqrt{(\varepsilon/(4C_0))^2-t^2}\ dt=(\varepsilon/(4C_0))^2\int_{-1}^1\sqrt{1-t^2}\ dt=\frac{\varepsilon^2}{C_0'},$$

thus

$$\int_{\mathbb{T}}|\omega_{\tau_*}^a|\ da\geq\frac{\sharp I\varepsilon^2}{C_0'}.$$

Comparing with (149), we obtain the upper bound

$$\sharp I\leq C_L\frac{|\log\varepsilon|}{\varepsilon}. \tag{150}$$

Applying Lemma 5.2 to the family of closed disks $(\bar{D}(a_i,5\varepsilon/(4C_0)))_{i\in I}$, there exists a family of closed pairwise disjoint disks $(\bar{D}(b_j,r_j))_{j\in J}$ such that

$$\sharp J\leq\sharp I\leq C_L\frac{|\log\varepsilon|}{\varepsilon},\qquad \cup_{i\in I}\ \bar{D}(a_i,5\varepsilon/(4C_0))\subset\cup_{j\in J}\bar{D}(b_j,r_j)$$

and, by (150),

$$\sum_{j\in J}r_j\leq\sum_{i\in I}5\varepsilon/(4C_0)=\sharp I\times 5\varepsilon/(4C_0)\leq C_L|\log\varepsilon|. \tag{151}$$

By construction, we have therefore localized S in disjoint closed cylinders :

$$S=\{|v_{\varepsilon,n}|\leq 1/2\}\subset\mathring{\cup}_{j\in J}\bar{C}(b_j,r_j), \tag{152}$$

which concludes the proof of Lemma 3. $\square$

6 Proofs of Proposition 3, Corollary 1 and Lemma 5

6.1 Proof of Proposition 3 : the speed is bounded

We give here the proof of Proposition 3 : the speed $c_{\varepsilon,n}$ is bounded for $0 < \varepsilon < \varepsilon_0(L)$ and $n \geq C_L|\log\varepsilon|^2$. We first recall the Besicovitch Covering Theorem.

Theorem 5. *Let E be a subset of $\mathbb{R}^N$ and let $r : E \to \mathbb{R}$ be a positive bounded function defined on E. Then one can choose an at most countable family of points $(x_i)_{i\in I}$ in E such that*

(i) $E \subset \cup_{i\in I}\bar{B}(x_i, r(x_i))$,

(ii) the balls $\bar{B}(x_i, \frac{1}{3}r(x_i))$ *are mutually disjoint,*

(iii) the balls $\bar{B}(x_i, r(x_i))_{i\in I}$ *can be distributed in at most $\zeta(N)$ families of disjoint closed balls, with $\zeta(N)$ depending only on N.*

Strategy of the proof of Proposition 3. The proof is based on Pohozaev identity (Step 1). The question is then to find a cylinder (or more) such that the momentum is large enough on this cylinder (Step 2) and $\int_{\check{C}(a,R)}\langle Ju_{\varepsilon,n}, \xi\rangle$ is close to the momentum on this cylinder. This introduces boundary terms when integrating by parts, that we will have to control (Step 4). We have also to bound the right-hand side of the Pohozaev identity. We can not use a too large cylinder (or too many) since the energy diverges as $n \to +\infty$. We will then have to bound the energy in some "small" cylinders (Step 5) : the lower bound given in Lemma 4.4 will be useful here. The estimates for the boundary terms will be established through an averaging argument, thus we will need to dilate a little the cylinder(s), and then to choose a suitable radius (Step 6). To conclude, note that one cylinder will not be enough and thus we will be compelled to work with many of them. In order to control the overlapping of these cylinders, we will make use of the Besicovitch Theorem.

Step 1: Pohozaev type identity. The following Pohozaev type identity holds for the solution $u_{\varepsilon,n}$ of (9). Let $C(a,R)$ be a cylinder, with $a \in D_n$ and $R > 0$, then

$$\int_{\check{C}(a,R)} |\partial_1 u_{\varepsilon,n}|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\ dx - \frac{c_{\varepsilon,n}}{2}|\log\varepsilon|\int_{\check{C}(a,R)}\langle Ju_{\varepsilon,n}, \xi\rangle$$
$$= \frac{1}{2}\int_{\mathbb{T}\times\partial(D(a,R)\cap D_n)}((x_2,x_3)-a)\cdot\nu\Big[|\nabla_\top u_{\varepsilon,n}|^2 - \Big|\frac{\partial u_{\varepsilon,n}}{\partial\nu}\Big|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\Big]. \tag{153}$$

For the proof, multiply, as for Pohozaev identity, the equation by $x_2\partial_2 u_{\varepsilon,n} + x_3\partial_3 u_{\varepsilon,n}$ and integrate by parts over $\check{C}(a,R)$ (note that we do not need an identification $\mathbb{R}/(2\pi\mathbb{Z}) \simeq [0, 2\pi)$ since the Pohozaev multiplier is 2π-periodic in the x_1 variable).

Step 2: Localizing the momentum of $v_{\varepsilon,n}$. We now estimate the contribution outside the cylinders $(\bar{C}(b_j, r_j))_{j\in J}$ given by Lemma 3 for the integral for the momentum of $v_{\varepsilon,n}$. We claim that, for any measurable set $\omega \subset G := D_n \setminus \mathring{\cup}_{j\in J}\bar{D}(b_j, r_j)$, we have

$$\Big|\int_\omega (iv_{\varepsilon,n}, \partial_1 v_{\varepsilon,n})\ dx\Big| \leq C_L\varepsilon|\log\varepsilon|,$$

for $n \geq C_L|\log\varepsilon|$, with C_L depending only on L.

First, notice that since the closed disks $(\bar{D}(b_j, r_j))_{j\in J}$ are pairwise disjoint, for $0 < \varepsilon < \varepsilon_0(L)$ and $n \geq (C_L+1)|\log\varepsilon|$ (C_L being the one in (151)),

$$G = D_n \setminus \mathring{\cup}_{j\in J}\bar{D}(b_j, r_j)$$

is connected. Hence, since $|v_{\varepsilon,n}| \geq 1/2$ outside $\Omega_n \setminus \mathring{\cup}_{j\in J}\bar{C}(b_j, r_j)$, for every $y \in G$,

$$\frac{v_{\varepsilon,n}(.,y)}{|v_{\varepsilon,n}(.,y)|} : \mathbb{T} \to \mathbb{S}^1$$

has a degree 0. Indeed, there exists at least one point, denoted y_*, in $\partial D_n \setminus \mathring{\cup}_{j\in J}\bar{D}(b_j, r_j)$, for otherwise, since the disks are pairwise disjoint, $r_j \geq n$ for at least one $j \in J$, which contradicts (151) if $n \geq (C_L+1)|\log\varepsilon|$. Consider $y \in G$. One can connect y to y_*, which gives rise to an homotopy from

$$\frac{v_{\varepsilon,n}(.,y)}{|v_{\varepsilon,n}(.,y)|} : \mathbb{T} \to \mathbb{S}^1 \qquad \text{to} \qquad \frac{v_{\varepsilon,n}(.,y_*)}{|v_{\varepsilon,n}(.,y_*)|} : \mathbb{T} \to \mathbb{S}^1,$$

which is constant (of value $\frac{y_*}{|y_*|}$) in view of the boundary condition, and then has degree 0. Therefore, one may write for $y \in G$

$$v_{\varepsilon,n}(.,y) = \rho(.,y)\exp(i\varphi(.,y)) \qquad \text{on } \mathbb{T}, \tag{154}$$

where $\rho(.,y) \geq 1/2$ and $\varphi(.,y) \in \mathbb{R}$ are lipschitz maps defined on $\mathbb{T}$ (the periodicity of φ comes from the fact that it has degree 0). We can not write (154) in the whole $\Omega_n \setminus \mathring{\cup}_{j\in J}\bar{C}(b_j, r_j)$ since $v_{\varepsilon,n}$ is expected to have a non-zero degree around (at least) one cylinder (in the (x_2, x_3) variables). Let $\omega \subset G = D_n \setminus \mathring{\cup}_{j\in J}\bar{D}(b_j, r_j)$ be measurable. Since $(iv_{\varepsilon,n}(.,y), \partial_1 v_{\varepsilon,n}(.,y)) = \rho(.,y)^2\partial_1\varphi(.,y)$ for $y \in \omega$, it follows that

$$\int_{\mathbb{T}\times\omega} (iv_{\varepsilon,n}, \partial_1 v_{\varepsilon,n})\, dx = \int_\omega \int_{\mathbb{T}} \rho(.,y)^2 \partial_1\varphi(.,y)\, dx_1 dy = \int_\omega \int_{\mathbb{T}} \Big(\rho(.,y)^2 - 1\Big)\partial_1\varphi(.,y)\, dx_1 dy$$

(since $\varphi(.,y)$ is well-defined on the torus, i.e. 2π-periodic). Hence, by Cauchy-Schwarz and (145)

$$\Big|\int_{\mathbb{T}\times\omega} (iv_{\varepsilon,n}, \partial_1 v_{\varepsilon,n})\, dx\Big| \leq 8\varepsilon\Big(\int_{\mathbb{T}\times\omega} \frac{(1-|v_{\varepsilon,n}|^2)^2}{4\varepsilon^2}\Big)^{1/2}\Big(\int_{\mathbb{T}\times\omega} |\partial_1 v_{\varepsilon,n}|^2\Big)^{1/2} \leq C_L\varepsilon|\log\varepsilon|,$$

which concludes the proof of the claim. □

Step 3: Going back to the Pohozaev identity. We infer from the Pohozaev identity of Step 1

$$|c_{\varepsilon,n}| \cdot \Big|\sum_{l=1}^q \int_{\omega_\lambda^l} \langle Ju_{\varepsilon,n}, \xi\rangle\Big| \leq C_L + \frac{2\zeta}{|\log\varepsilon|}\frac{d}{d\lambda}\sum_{j\in J}\int_{\check{C}(b_j,\lambda r_j)} e_\varepsilon(u_{\varepsilon,n}) + C_L\int_{\partial\Omega_n} e_\varepsilon(u_{\varepsilon,n}). \tag{155}$$

We apply the Besicovitch Covering Theorem to the family $(\bar{D}(b_j, 3r_j))_{j\in J}$. It provides us a partition $(J_l)_{1\leq l\leq q}$ of $\tilde{J} \subset J$, with $q \leq \zeta$ (ζ being an absolute integer), such that

$$\cup_{j\in J}\bar{D}(b_j, 3r_j) \subset \cup_{j\in\tilde{J}}\bar{D}(b_j, 3r_j) \tag{156}$$

and for $1 \leq l \leq q$, the disks $\bar{D}(b_j, 3r_j)$, $j \in J_l$, are pairwise disjoint. Next, for every $1 \leq \lambda \leq 3$, we apply Step 1 on each $C(b_j, \lambda r_j)$ and deduce by summing over $j \in J_l$ (since the disks $\bar{D}(b_j, \lambda r_j)$, $j \in J_l$, are pairwise disjoint for $1 \leq \lambda \leq 3$ and $1 \leq l \leq q$) that, denoting $\omega_\lambda^l := \cup_{j\in J_l} D(b_j, \lambda r_j)$,

$$\begin{aligned} c_{\varepsilon,n}|\log\varepsilon| \sum_{l=1}^q \int_{\omega_\lambda^l} \langle Ju_{\varepsilon,n}, \xi\rangle &= 2\sum_{l=1}^q \int_{\omega_\lambda^l} |\partial_1 u_{\varepsilon,n}|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\, dx \\ &- \sum_{l=1}^q \sum_{j\in J_l} \int_{\partial\check{C}(b_j,\lambda r_j)} ((x_2,x_3) - b_j)\cdot\nu\Big[|\nabla_\top u_{\varepsilon,n}|^2 - \Big|\frac{\partial u_{\varepsilon,n}}{\partial\nu}\Big|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\Big]. \end{aligned} \tag{157}$$

Since $q \le \zeta$, we deduce from (20) that the first sum in the right-hand side of (157) satisfies

$$2\sum_{l=1}^{q}\int_{\omega_\lambda^l}|\partial_1 u_{\varepsilon,n}|^2+\frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\,dx \le 2q\int_{\cup_{j\in J}D(b_j,3r_j)}|\partial_1 u_{\varepsilon,n}|^2+\frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\,dx$$
$$\le 2\zeta C_L|\log\varepsilon|. \tag{158}$$

Concerning the last sum in (157), we note that, there, $|((x_2,x_3)-b_j)\cdot\nu| \le r_j$ and

$$\Big||\nabla_\top u_{\varepsilon,n}|^2-\Big|\frac{\partial u_{\varepsilon,n}}{\partial\nu}\Big|^2+\frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\Big| \le 2e_\varepsilon(u_{\varepsilon,n}),$$

thus, since the disks $D(b_j,\lambda r_j)$, $j\in J_l$, are pairwise disjoint for $1\le l\le q$ and $1\le\lambda\le 3$,

$$\Big|\sum_{l=1}^{q}\sum_{j\in J_l}\int_{\partial\check C(b_j,\lambda r_j)}((x_2,x_3)-b_j)\cdot\nu\Big[|\nabla_\top u_{\varepsilon,n}|^2-\Big|\frac{\partial u_{\varepsilon,n}}{\partial\nu}\Big|^2+\frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\Big]\Big|$$
$$\le 2\zeta\sum_{j\in J}r_j\int_{\partial\check C(b_j,\lambda r_j)}e_\varepsilon(u_{\varepsilon,n})$$
$$\le 2\zeta\frac{d}{d\lambda}\sum_{j\in J}\int_{\check C(b_j,\lambda r_j)}e_\varepsilon(u_{\varepsilon,n})+2\zeta\sum_{j\in J}r_j\int_{\partial\Omega_n\cap C(b_j,\lambda r_j)}e_\varepsilon(u_{\varepsilon,n})$$
$$\le 2\zeta\frac{d}{d\lambda}\sum_{j\in J}\int_{\check C(b_j,\lambda r_j)}e_\varepsilon(u_{\varepsilon,n})+C_L|\log\varepsilon|\int_{\partial\Omega_n}e_\varepsilon(u_{\varepsilon,n}) \tag{159}$$

by (151). Combining (158) and (159) with (157), we are led to (155). □

Step 4: Control for the momentum and the boundary terms. We prove that

$$\Big|\sum_{l=1}^{q}\int_{\omega_\lambda^l}\langle Ju_{\varepsilon,n},\xi\rangle\Big| \ge \Big|\sum_{l=1}^{q}\int_{\omega_\lambda^l\setminus G}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big|-C_L\varepsilon|\log\varepsilon|-4\sqrt{\varepsilon}\frac{d}{d\lambda}\sum_{j\in J}\int_{\check C(b_j,\lambda r_j)}f_\varepsilon \tag{160}$$

and

$$n\int_{\mathbb{T}\times\partial D_n}e_\varepsilon(u_{\varepsilon,n}) \le C_L|\log\varepsilon|+C_L|c_{\varepsilon,n}|\cdot|\log\varepsilon|. \tag{161}$$

First, note that integration by parts yields

$$\int_{\check C(b_j,\lambda r_j)}\langle Ju_{\varepsilon,n},\xi\rangle=\int_{\check C(b_j,\lambda r_j)}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})-\frac{1}{2}\int_{\mathbb{T}\times\partial(D(b_j,\lambda r_j)\cap D_n)}((x_2,x_3)-b_j)\cdot\nu(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n}),$$

and therefore (the disks $D(b_j,\lambda r_j)$, $j\in J_l$, are pairwise disjoint for $1\le\lambda\le 3$ and $1\le l\le q$)

$$\Big|\sum_{l=1}^{q}\int_{\omega_\lambda^l}\langle Ju_{\varepsilon,n},\xi\rangle\Big| \ge \Big|\sum_{l=1}^{q}\int_{\omega_\lambda^l}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big|$$
$$-\frac{1}{2}\Big|\sum_{l=1}^{q}\sum_{j\in J_l}\int_{\partial\check C(b_j,\lambda r_j)}((x_2,x_3)-b_j)\cdot\nu(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big|. \tag{162}$$

As in the proof of Lemma 2, we have

$$\Big|\int_{\partial\check C(b_j,\lambda r_j)}((x_2,x_3)-b_j)\cdot\nu\ (iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})-\int_{\partial\check C(b_j,\lambda r_j)}((x_2,x_3)-b_j)\cdot\nu\ (i\tilde u,\partial_1\tilde u)\Big| \tag{163}$$
$$=\Big|\int_{\partial\check C(b_j,\lambda r_j)}((x_2,x_3)-b_j)\cdot\nu(i(u_{\varepsilon,n}-\tilde u),\partial_1(u_{\varepsilon,n}+\tilde u))\Big| \le \varepsilon r_j\int_{\partial\check C(b_j,\lambda r_j)}f_\varepsilon=\varepsilon\frac{d}{d\lambda}\int_{\check C(b_j,\lambda r_j)}f_\varepsilon,$$

where

$$f_\varepsilon := |\partial_1 u_{\varepsilon,n}|^2 + |\partial_1 v_{\varepsilon,n}|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2} + \frac{(1-|v_{\varepsilon,n}|^2)^2}{2\varepsilon^2} + \frac{|v_{\varepsilon,n}-\tilde u|^2}{\varepsilon},$$

since by definition of g, $f_\varepsilon = 0$ on $\partial\Omega_n$, and similarly

$$\Big|\int_{\partial\check C(b_j,\lambda r_j)} ((x_2,x_3)-b_j)\cdot\nu(i\tilde u,\partial_1\tilde u) - \int_{\partial\check C(b_j,\lambda r_j)} ((x_2,x_3)-b_j)\cdot\nu(iv_{\varepsilon,n},\partial_1 v_{\varepsilon,n})\Big|$$
$$\leq \sqrt{\varepsilon} r_j \int_{\partial\check C(b_j,\lambda r_j)} f_\varepsilon = \sqrt{\varepsilon}\frac{d}{d\lambda}\int_{\check C(b_j,\lambda r_j)} f_\varepsilon. \tag{164}$$

Moreover, we also have (since $|v_{\varepsilon,n}|\geq 1/2$ on $\partial\check C(b_j,\lambda r_j)$, we may write $v_{\varepsilon,n}(.,y) = \rho(.,y)e^{i\varphi(.,y)}$ on $\mathbb{T}$ for $y\in\partial\check D(b_j,\lambda r_j)$ and for a lipschitz φ, x_1-periodic), as in Step 2, by Cauchy-Schwarz and (145)

$$\Big|\int_{\partial\check C(b_j,\lambda r_j)} ((x_2,x_3)-b_j)\cdot\nu\ (iv_{\varepsilon,n},\partial_1 v_{\varepsilon,n})\Big| = \Big|\int_{\partial(D(b_j,\lambda r_j)\cap D_n)}\int_{\mathbb{T}} \rho^2(.,y)\partial_1\varphi(.,y)\ dx_1 dy\Big|$$
$$= \varepsilon\Big|\int_{\partial(D(b_j,\lambda r_j)\cap D_n)}\int_{\mathbb{T}} \frac{(\rho^2(.,y)-1)}{\varepsilon}\partial_1\varphi(.,y)\ dx_1 dy\Big|$$
$$\leq 2\varepsilon\frac{d}{d\lambda}\int_{\check C(b_j,\lambda r_j)} f_\varepsilon. \tag{165}$$

Combining inequalities (163), (164) and (165) with (162) implies

$$\Big|\sum_{l=1}^q\int_{\omega_\lambda^l}\langle Ju_{\varepsilon,n},\xi\rangle\Big| \geq \Big|\sum_{l=1}^q\int_{\omega_\lambda^l}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big| - 4\sqrt{\varepsilon}\frac{d}{d\lambda}\sum_{j\in J}\int_{\check C(b_j,\lambda r_j)} f_\varepsilon. \tag{166}$$

Next, notice that

$$\Big|\sum_{l=1}^q\int_{\omega_\lambda^l}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big| \geq \Big|\sum_{l=1}^q\int_{\omega_\lambda^l\setminus G}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big| - \sum_{l=1}^q\Big|\int_{\omega_\lambda^l\cap G}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big|, \tag{167}$$

where, we recall, $G = D_n\setminus \overset{\circ}{\cup}_{j\in J} D(b_j,r_j)$ is the set where the momentum is $\simeq 0$. From Step 2 with

$$\omega := \omega_\lambda^l \setminus \Big(\cup_{j\in J} D(b_j,r_j)\Big) \subset G$$

for $1\leq l\leq q$, we have

$$\Big|\int_{\omega_\lambda^l\setminus G}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big| \leq C_L\varepsilon|\log\varepsilon|,$$

thus summing these inequalities for $1\leq l\leq q\leq\zeta$ yields

$$\sum_{l=1}^q\Big|\int_{\omega_\lambda^l\setminus G}(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\Big| \leq C_L\zeta\varepsilon|\log\varepsilon|. \tag{168}$$

Combining (166), (167) and (168) gives (160).

Concerning the boundary energy, we know that $u_{\varepsilon,n} = g = e^{i\theta}$ on $\partial\Omega_n$, so $|\nabla_\top g|^2 = n^{-2}$ and

$$2\int_{\partial\Omega_n} e_\varepsilon(u_{\varepsilon,n}) = \int_{\partial\Omega_n}\frac{1}{n^2} + \Big|\frac{\partial u_{\varepsilon,n}}{\partial\nu}\Big|^2 = \frac{4\pi^2}{n} + \int_{\partial\Omega_n}\Big|\frac{\partial u_{\varepsilon,n}}{\partial\nu}\Big|^2. \tag{169}$$

Next, by the Pohozaev identity of Step 1 on $\Omega_n = C_n(0)$ (recalling $p(u_{\varepsilon,n}) = \int_{\Omega_n}\langle Ju_{\varepsilon,n},\xi\rangle = 2\pi^2 L^2$)

$$\int_{\Omega_n}|\partial_1 u_{\varepsilon,n}|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\ dx - \pi^2L^2 c_{\varepsilon,n}|\log\varepsilon| = 2\pi^2 - \frac{n}{2}\int_{\mathbb{T}\times\partial D_n}\Big|\frac{\partial u_{\varepsilon,n}}{\partial\nu}\Big|^2. \tag{170}$$

From (20), (169) and (170), we infer (161). □

Step 5: Upper bound for the energy on the cylinders. We claim that

$$E_\varepsilon(u_{\varepsilon,n}, \cup_{j\in J}\check{C}(b_j, 3r_j)) \le C_L|\log\varepsilon|. \tag{171}$$

Applying Lemma 4.4 with $H = \bar{D}_n \cap \cup_{j\in J}\bar{D}(b_j, 3r_j)$ yields

$$\frac{1}{2}\int_{\mathbb{T}\times(D_n\setminus H)} |\nabla_{2,3}u_{\varepsilon,n}|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2} \ge 2\pi^2\log n + 2\pi^2(1-t_*^2)|\log\varepsilon| - 2\pi^2 t_*^2\log(|H|) - C,$$

where C is independent of ε, n and H and $t_* := \sqrt{1+\left(\frac{\pi\varepsilon}{2\sqrt{2}|H|}\right)^2} - \frac{\pi\varepsilon}{2\sqrt{2}|H|} \in [0,1]$. We then infer

$$E_\varepsilon(u_{\varepsilon,n}, \Omega_n \setminus \cup_{j\in J}\cup_{j\in J}\check{C}(b_j, 3r_j)) \ge 2\pi^2\log n - 2\pi^2\log(|\log\varepsilon|) - C,$$

since from (151), we have

$$|H| \le 3\sum_{j\in J} r_j \le C_L|\log\varepsilon|,$$

and (171) follows from (19).

Step 6: Choice of λ. Notice that, since $G = D_n \setminus \cup_{j\in J}D(b_j, r_j)$,

$$\omega_\lambda^l \setminus G = \cup_{j\in J_l}D(b_j, r_j).$$

Hence, the disks $\bar{D}(b_j, r_j)$ for $j\in J$ being mutually disjoint,

$$\sum_{l=1}^q \int_{\mathbb{T}\times\omega_\lambda^l\setminus G}(iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}) = \int_{\cup_{j\in\tilde{J}}\check{C}(b_j,r_j)}(iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}).$$

Moreover, in view of the constraint $p(u_{\varepsilon,n}) = 2\pi^2L^2$, Lemma 2 and Step 2 with $\omega = G$, we have, denoting $J_0 := J\setminus\tilde{J}$,

$$\begin{aligned}\int_{\cup_{j\in\tilde{J}}D(b_j,r_j)}(iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}) + \int_{\cup_{j\in J_0}D(b_j,r_j)}(iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}) &= \int_{\cup_{j\in J}D(b_j,r_j)}(iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n})\\ &\ge 2\pi^2L^2 - C_L\varepsilon|\log\varepsilon| \ge \pi^2L^2 > 0,\end{aligned} \tag{172}$$

for $\varepsilon > 0$ sufficiently small. Assume

$$\int_{\cup_{j\in\tilde{J}}D(b_j,r_j)}(iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}) \ge \frac{1}{2}\pi^2L^2 > 0, \tag{173}$$

then (160) rewrites

$$\Big|\sum_{l=1}^q \int_{\omega_\lambda^l}\langle Ju_{\varepsilon,n}, \xi\rangle\Big| \ge \frac{1}{2}\pi^2L^2 - C_L\varepsilon|\log\varepsilon| - \sqrt{\varepsilon}\frac{d}{d\lambda}\sum_{j\in J}\int_{\check{C}(b_j,\lambda r_j)} f_\varepsilon. \tag{174}$$

Now, we choose $\lambda\in[1,3]$. From (20) and (145), we know that

$$\int_1^3 \frac{d}{d\lambda}\sum_{j\in J}\int_{\check{C}(b_j,\lambda r_j)} f_\varepsilon\, d\lambda \le \zeta\int_{\cup_{j\in J}\check{C}(b_j,3r_j)} f_\varepsilon \le \zeta C_L|\log\varepsilon|. \tag{175}$$

Moreover, by Step 5, we have

$$\int_1^3 \frac{d}{d\lambda} \sum_{j\in J} \int_{\check{C}(b_j,\lambda r_j)} e_\varepsilon(u_{\varepsilon,n})\, d\lambda \leq \zeta \int_{\cup_{j\in J}\check{C}(b_j,3r_j)} e_\varepsilon(u_{\varepsilon,n}) \leq C_L|\log\varepsilon|. \tag{176}$$

Combining (175) and (176) and the mean-value formula, we deduce that there exists $\lambda \in [1,3]$ (depending on ε, n and L) such that

$$\frac{d}{d\lambda} \sum_{j\in J} \int_{\check{C}(b_j,\lambda r_j)} f_\varepsilon + \frac{d}{d\lambda} \sum_{j\in J} \int_{\check{C}(b_j,\lambda r_j)} e_\varepsilon(u_{\varepsilon,n}) \leq C_L|\log\varepsilon|. \tag{177}$$

In particular, for $\varepsilon > 0$ sufficiently small, (174) implies

$$\Big|\sum_{l=1}^q \int_{\omega_\lambda^l} \langle Ju_{\varepsilon,n}, \xi\rangle\Big| \geq \frac{1}{4}\pi^2 L^2. \tag{178}$$

Inserting (161), (174) and (178) into (155) yields

$$\frac{\pi^2 L^2}{4}|c_{\varepsilon,n}| \leq |c_{\varepsilon,n}| \cdot \Big|\sum_{l=1}^q \int_{\omega_\lambda^l} \langle Ju_{\varepsilon,n}, \xi\rangle\Big| \leq C_L + C_L\frac{|\log\varepsilon|}{n}(1+|c_{\varepsilon,n}|). \tag{179}$$

If $n \geq C_L|\log\varepsilon|^2$ and ε is small enough so that $C_L\frac{|\log\varepsilon|}{n} \leq \frac{1}{8}\pi^2L^2$, then

$$\frac{\pi^2L^2}{4}|c_{\varepsilon,n}| \leq C_L + \frac{1}{8}\pi^2L^2|c_{\varepsilon,n}|,$$

which yields the desired estimate (for $L > 0$)

$$|c_{\varepsilon,n}| \leq K(L)$$

in the case where (173) is satisfied. If it is not, from (172), we deduce that therefore

$$\int_{\cup_{j\in J_0} D(b_j,r_j)} (iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}) \geq \frac{1}{2}\pi^2L^2 > 0, \tag{180}$$

which means that the cylinders $\check{C}(b_j,r_j)$ for $j \in J_0$ concentrate a "good part" of the momentum. We may then "forget" the other cylinders and argue as previously (Steps 2, 3 and 4) with the new collection of disks $(D(b_j,3r_j))_{j\in J_0}$ instead of $(D(b_j,3r_j))_{j\in J}$. Indeed, when we have applied the Besicovitch Theorem, we have obtained a partition (J_l), $1 \leq l \leq q$, of $\tilde{J}$ such that

$$\cup_{j\in J}\bar{D}(b_j,3r_j) \subset \cup_{j\in\tilde{J}}\bar{D}(b_j,3r_j).$$

Since

$$\cup_{j\in J}\bar{D}(b_j,r_j) \subset \cup_{j\in J}\bar{D}(b_j,3r_j) \subset \cup_{j\in\tilde{J}}\bar{D}(b_j,3r_j),$$

this induces a partition of J_0 in

$$J_0^l := \{j \in J_0, \bar{D}(b_j,r_j) \subset \cup_{\tilde{j}\in J_l}\bar{D}(b_{\tilde{j}},3r_{\tilde{j}})\}$$

for $1 \leq l \leq q \leq \zeta$ such that the disks $\bar{D}(b_j,3r_j)$, $j \in J_0^l$, are mutually disjoint. We follow then Steps 2, 3 and 4 with this collection satisfying (180) and with controlled overlapping. The proof of Proposition 3 is then complete. □

6.2 Proof of Corollary 1 : fine localization of the singular set

We now apply the arguments used in subsection 5.2 for the (rough) location of the singular set of $v_{\varepsilon,n}$ to $u_{\varepsilon,n}$. This enables us to exhibit a family of cylinders, for which the sum of the radii is not too large, and that concentrate the $|\log\varepsilon|$ term of the energy.

First, we apply the results of the previous subsection 5.2 to $u_{\varepsilon,n}$. This is possible since it uses the upper bounds (19) and (20), together with the estimate on the gradient (4)

$$|\nabla u_{\varepsilon,n}| \leq \frac{C_L}{\varepsilon}.$$

The consequence is that there exists a family of disks $(D(b_j, r_j))_{j\in J}$ such that

$$\sharp J \leq C_L \frac{|\log\varepsilon|}{\varepsilon}, \qquad \sum_{j\in J} r_j \leq C_L |\log\varepsilon|, \tag{181}$$

$$S^n_\varepsilon \subset \cup_{j\in J}\check{C}(b_j, r_j) \qquad \text{and} \qquad E_\varepsilon(u_{\varepsilon,n}, \cup_{j\in J}\check{C}(b_j, r_j)) \leq C_L |\log\varepsilon|. \tag{182}$$

We may therefore apply the Clearing-Out result of Theorem 4 to assert the existence of $R_0 > 0$ and $\eta > 0$ (independent of ε, $n \geq C_L|\log\varepsilon|^2$ and of the b_j's and r_j's) such that for each $x \in S^n_\varepsilon$,

$$E_\varepsilon(u_{\varepsilon,n}, B(x, R_0) \cap \Omega_n) \geq \eta|\log\varepsilon|. \tag{183}$$

Applying Lemma 4.4 with $H = \cup_{j\in J}\check{D}(b_j, r_j + R_0)$, we obtain

$$\frac{1}{2}\int_{\mathbb{T}\times(D_n\setminus H)} |\nabla u_{\varepsilon,n}|^2 + \frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2} \geq 2\pi^2\log n + 2\pi^2(1-t_*^2)|\log\varepsilon| - 2\pi^2 t_*^2 \log(|H|) - C,$$

for C independent of ε, n and H and $t_* \in [0, 1]$. By (181), $|H| \leq \frac{|\log\varepsilon|}{\varepsilon}$, thus, using (19),

$$E_\varepsilon(u_{\varepsilon,n}, \cup_{j\in J}\check{C}(b_j, r_j + R_0)) \leq C_L|\log\varepsilon|. \tag{184}$$

Therefore, S^n_ε being covered by the balls $B(y, 5R_0)$, $y \in S^n_\varepsilon$, it follows from Vitali's covering theorem that there exists an at most countable family of points $(y_i)_{i\in I}$ in S^n_ε such that

$$S^n_\varepsilon \subset \cup_{i\in I} B(y_i, 5R_0)$$

and the balls $B(y_i, R_0)$ are mutually disjoint. As a consequence, from (183) and (184), we deduce

$$\sharp I \leq \frac{C_L}{\eta} := l. \tag{185}$$

We then proceed as in Step 1 of the proof of Theorem 4 in [BOS] (Appendix C) to conclude to the existence of cylinders $C(a_i, R_0)$ $(1 \leq i \leq q \leq l)$ (with a different R_0 than before) such that

$$S^n_\varepsilon \subset \cup_{i=1}^q C(a_i, R_0)$$

and the cylinders $C(a_i, 8R_0)$ are mutually disjoint. We are then left with (27). We apply Lemma 4.4 with $H = \bar{D}_n \cap \cup_{i=1}^q \bar{D}(a_i, R_0)$, as for (184), to deduce

$$E_\varepsilon(u_{\varepsilon,n}, \mathbb{T}\times(D_n \setminus H)) \geq 2\pi^2\log n - C_L,$$

since, by (185),

$$|H| \leq qR_0 \leq lR_0 \leq C_L.$$

This implies (28) by (19). The proof of Corollary 1 is complete. $\square$

6.3 Proof of Lemma 5 : defining the limiting current

The proof follows the one of Lemma 5 in [BOS]. Arguing as in Lemma 3.3 of [BOS], we have the following lemma.

Lemma 6.1. *Let $M_0 > 0$ and $R > 0$ and $X := \{u \in H^1(C_{4R}, \mathbb{C}),\ |u| \geq 1/2$ in $C_{4R} \setminus C_R\}$. Then, for any $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(M_0, R, \delta) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $u \in X$ satisfying $E_\varepsilon(u) \leq M_0|\log \varepsilon|$, there exists a 1-dimensional integral current T without boundary supported in C_R such that*

$$\|Ju - \pi T\|_{[\mathcal{C}_c^{0,1}(C_{4R})]^*} \leq \delta \qquad \text{and} \qquad \mathbf{M}(T) \leq \frac{E_\varepsilon(u)}{\pi|\log \varepsilon|} + \delta.$$

Proof of Lemma 5. First, we extend $\tilde{u}_{\varepsilon,n}$ by $e^{i\theta}$ outside Ω_n. The energy of the extension on $C(a_i, 8R_0) \setminus C(a_i, R_0)$ is less than or equal to $2\pi^2 \log(\frac{n+8R_0}{n}) \leq C_L$. We then apply Lemma 6.1 to this extension of $\tilde{u}_{\varepsilon,n}$ on each cylinder $C(a_i, 4R_0)$. This provides us, for $0 < \varepsilon < \varepsilon_0(L)$ sufficiently small a 1-dimensional integral current T without boundary supported in $C(a_i, R_0)$ such that

$$\|J\tilde{u}_{\varepsilon,n} - \pi T_i\|_{[\mathcal{C}_c^{0,1}(C(a_i,4R_0)]^*} \leq r(\varepsilon) \qquad \text{and} \qquad \mathbf{M}(T_i) \leq \frac{E_\varepsilon(u_{\varepsilon,n}, C(a_i, 4R_0))}{\pi|\log \varepsilon|} + r(\varepsilon).$$

We let $T := \sum_{i=1}^q T_i$. Then, we have

$$\mathbf{M}(T) \leq \frac{E_\varepsilon(u_{\varepsilon,n}, \cup_{i=1}^q C(a_i, 4R_0))}{\pi|\log \varepsilon|} + r(\varepsilon),$$

which is $iii)$, and $i)$ follows easily. We are then left with $ii)$. For $1 \leq i \leq q$, let $\xi_i : C(a_i, 2R_0) \to \Lambda^2\mathbb{R}^3$ be a smooth map compactly supported such that $\xi_i \equiv (a_i)_2\ dx_1 \wedge dx_2 + (a_i)_3\ dx_1 \wedge dx_3$ in $C(a_i, R_0)$ and $\|\xi_i\|_{L^\infty(C(a_i,2R_0))} \leq C_L$. Then, since $J\tilde{u}_{\varepsilon,n}$ and $T_{\varepsilon,n}$ are supported in $\cup_{i=1}^q C(a_i, R_0)$, we infer from the equality $p(u_{\varepsilon,n}) = \int_{\Omega_n} \langle Ju_{\varepsilon,n}, \xi\rangle$ that

$$\big|p(u_{\varepsilon,n}) - \mathcal{F}(T_{\varepsilon,n})\big| \leq \Big|\int_{\Omega_n} \langle Ju_{\varepsilon,n} - J\tilde{u}_{\varepsilon,n}, \xi\rangle\Big| + \sum_{i=1}^q \Big|\int_{C(a_i,2R_0)} \langle J\tilde{u}_{\varepsilon,n} - \pi T_{\varepsilon,n}, \xi\rangle\Big|. \tag{186}$$

For the second term, we write, by construction of ξ_i,

$$\sum_{i=1}^q \Big|\int_{C(a_i,2R_0)} \langle J\tilde{u}_{\varepsilon,n} - \pi T_i, \xi\rangle\Big| \leq \sum_{i=1}^q \Big|\int_{C(a_i,2R_0)} \langle J\tilde{u}_{\varepsilon,n} - \pi T_i, \xi - \xi_i\rangle\Big| \leq qC_L r(\varepsilon) = r(\varepsilon), \tag{187}$$

since $\|\xi - \xi_i\|_{\mathcal{C}_c^{0,1}(C(a_i,2R_0))} \leq C_L$ (but $\|\xi\|_{\mathcal{C}_c^{0,1}(C(a_i,2R_0))} \to +\infty$ if $\|a_i\| \to +\infty$ as $n \to +\infty$) and $q \leq l$. Concerning the first term, we integrate by parts (note that $u_{\varepsilon,n} = \tilde{u}_{\varepsilon,n} = g = e^{i\theta}$ on $\partial\Omega_n$) to obtain

$$\Big|\int_{\Omega_n} \langle Ju_{\varepsilon,n} - J\tilde{u}_{\varepsilon,n}, \xi\rangle\Big| = \Big|\int_{\Omega_n} (iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}) - (i\tilde{u}_{\varepsilon,n}, \partial_1 \tilde{u}_{\varepsilon,n})\Big|. \tag{188}$$

It suffices then to write that, by Cauchy-Schwarz,

$$\begin{aligned}
\int_{\{|u_{\varepsilon,n}|\leq 1/2\}} \big|(iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n}) - (i\tilde{u}_{\varepsilon,n}, \partial_1 \tilde{u}_{\varepsilon,n})\big| &\leq 3\int_{\{|u_{\varepsilon,n}|\leq 1/2\}} \big|(iu_{\varepsilon,n}, \partial_1 u_{\varepsilon,n})\big| \\
&\leq C_L|\{|u_{\varepsilon,n}| \leq 1/2\}|^{1/2}\Big(\int_{\Omega_n} |\partial_1 u_{\varepsilon,n}|^2\Big)^{1/2} \qquad (189) \\
&\leq C_L\varepsilon\Big(\int_{\Omega_n} \frac{(1 - |u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\Big)^{1/2}\Big(\int_{\Omega_n} |\partial_1 u_{\varepsilon,n}|^2\Big)^{1/2} = r(\varepsilon)
\end{aligned}$$

by (20). Also, still by (20),

$$\int_{\{|u_{\varepsilon,n}|>1/2\}} \big|(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})-(i\tilde u_{\varepsilon,n},\partial_1\tilde u_{\varepsilon,n})\big| \le C_L\int_{\{|u_{\varepsilon,n}|>1/2\}}\Big|1-\frac{1}{|u_{\varepsilon,n}|^2}\Big|\cdot\big|(iu_{\varepsilon,n},\partial_1 u_{\varepsilon,n})\big| \tag{190}$$
$$\le C_L\varepsilon\Big(\int_{\Omega_n}\frac{(1-|u_{\varepsilon,n}|^2)^2}{2\varepsilon^2}\Big)^{1/2}\Big(\int_{\Omega_n}|\partial_1 u_{\varepsilon,n}|^2\Big)^{1/2}=r(\varepsilon).$$

Combining (187), (188), (189), (190) with (186) gives *ii*). We emphasize however that *i*) is stated with $J\tilde u_{\varepsilon,n}$ and not $Ju_{\varepsilon,n}$, since we do not know yet that these two jacobians are close globally in Ω_n (compare with Lemma 3.1 in [BOS]) since we do not have a bound $E_\varepsilon(u_{\varepsilon,n})\le M_0|\log\varepsilon|$. However, since $u_{\varepsilon,n}$ satisfies the local bound (28), we deduce (29) from Lemma 3.1 in [BOS]. □

7 Proof of Proposition 4 : the current tends to the helix

The proof of Proposition 4, where we prove that the integral current $T_{\varepsilon,n}$ is supported in a single cylinder and tends, up to a translation, to the helix $\vec{\mathcal{H}}_L$, is divided in several steps.

Step 1: We prove that $T_{\varepsilon,n}$ is close to an helix.

Lemma 7.1. *For every sequence ε_j and $n_j\ge C_L|\log\varepsilon_j|^2$, there exists a subsequence, still denoted ε_j and n_j, and a translation τ_j in $\mathbb{T}\times\mathbb{R}^2$ such that*

$$\tau_j(T_{\varepsilon_j,n_j})\to\vec{\mathcal{H}}_L \quad \textit{in } [\mathcal{C}_c^{0,1}(\mathbb{T}\times\mathbb{R}^2)]^* \quad \textit{as } j\to+\infty.$$

Proof of Lemma 7.1. We first note that, by Lemma 5, $T_{\varepsilon,n}$ is without boundary and satisfies

$$\mathbf{M}(T_{\varepsilon,n})\le 2\pi\sqrt{1+L^2}+r(\varepsilon) \quad\text{and}\quad |\mathcal{F}(T_{\varepsilon,n})-2\pi^2L^2|\le r(\varepsilon). \tag{191}$$

Therefore, from [F] (Theorem 4.2.17), there exists, up to a possible subsequence, a translation τ_j in $\mathbb{T}\times\mathbb{R}^2$ and a 1-dimensional integral current T, without boundary, such that

$$\tau_j(T_{\varepsilon_j,n_j})\to T \quad \text{in } [\mathcal{C}_c^{0,1}(\mathbb{T}\times\mathbb{R}^2)]^* \quad \text{as } j\to+\infty.$$

Passing to the limit in (191) yields

$$\mathbf{M}(T)\le 2\pi\sqrt{1+L^2} \quad\text{and}\quad \mathcal{F}(T)=2\pi^2L^2.$$

Moreover, in view of the boundary condition, $\langle J\tilde u_{\varepsilon,n}, dx_2\wedge dx_3\rangle=2\pi$. Since $J\tilde u_{\varepsilon,n}$ is supported in the cylinders $\check C(a_i,R_0)$ for $1\le i\le q\le l$ (l being independent of ε and n) with the cylinders $\check C(a_i,8R_0)$ mutually disjoint, we can construct a 2-form ζ in $\mathbb{T}\times\mathbb{R}^2$ (depending on the cylinders $C(a_i,R_0)$, $1\le i\le q$) such that ζ is supported in the cylinders $C(a_i,3R_0)$, $1\le i\le q$, $\zeta=dx_2\wedge dx_3$ in the cylinders $C(a_i,2R_0)$, $1\le i\le q$, and $\|\zeta\|_{\mathcal{C}_c^{0,1}(\mathbb{T}\times\mathbb{R}^2)}\le C_L$. Thus, by Lemma 5 *i*),

$$2\pi=\langle J\tilde u_{\varepsilon,n},dx_2\wedge dx_3\rangle=\langle J\tilde u_{\varepsilon,n},\zeta\rangle=\langle T_{\varepsilon,n},\zeta\rangle+r(\varepsilon)=\langle T,\zeta\rangle+r(\varepsilon),$$

since ζ is uniformly bounded in $\mathcal{C}_c^{0,1}(\mathbb{T}\times\mathbb{R}^2)$. Consequently,

$$Pr_1(T)=2\pi.$$

We may now apply Lemma 6 to obtain the existence of a translation t in $\mathbb{T}\times\mathbb{R}^2$ such that $t(T)=\vec{\mathcal{H}}_L$, and the proof of Lemma 7.1 is complete by replacing τ_j by $t\circ\tau_j$. □

From now on, we work for any sequence ε_j, $n_j \geq C_L|\log \varepsilon_j|^2$. From Step 1, we have extracted a subsequence, still denoted ε_j, n_j.

Step 2: We then prove (30), that is

there exist $R_0 > 0$, depending only on L, and $a \in \mathbb{T} \times \mathbb{R}^2$, depending on ε_j and n_j, such that

$$Supp(T_{\varepsilon_j,n_j}) \subset C(a, R_0).$$

We proceed as in the proof of Lemma 6 in [BOS], arguing by contradiction. By Step 1, (up to a subsequence) we have

$$\tau_j(T_{\varepsilon_j,n_j}) \to \vec{\mathcal{H}}_L \quad \text{ in } [\mathcal{C}_c^{0,1}(\mathbb{T} \times \mathbb{R}^2)]^* \quad \text{ as } j \to +\infty.$$

Let $a := \tau_j^{-1}(0)$. We may assume, relabelling the a_i's if necessary, that $a \in C(a_1, R_0)$. If, for some $1 < i_0 \leq q$, $S_{\varepsilon_j}^{n_j} \cap C(a_{i_0}, 8R_0) \neq \emptyset$, then by Step 1, Theorem 4 with $\sigma = 1/2$ and Lemma 5 *iii)*,

$$\mathbf{M}(\mathcal{H}_L) \leq \liminf_{j\to+\infty} \frac{E_{\varepsilon_j}(u_{\varepsilon_j,n_j}, C(a_1, 8R_0))}{\pi|\log \varepsilon_j|} \quad \text{ and } \quad 0 < \eta_{\sigma=1/2} \leq \liminf_{j\to+\infty} \frac{E_{\varepsilon_j}(u_{\varepsilon_j,n_j}, C(a_{i_0}, 8R_0))}{\pi|\log \varepsilon_j|},$$

thus

$$\mathbf{M}(\mathcal{H}_L) + \eta = 2\pi\sqrt{1+L^2} + \eta \leq \liminf_{j\to+\infty} \frac{E_{\varepsilon_j}(u_{\varepsilon_j,n_j}, \cup_{i=1}^q C(a_i, 8R_0))}{\pi|\log \varepsilon_j|} \leq 2\pi\sqrt{1+L^2},$$

by (27) in Corollary 1. This is a contradiction. □

Up to a translation of vector $\vec{e}_1 a_1$, we may assume that $a = a(\varepsilon_j, n_j) = (0, b)$, $b \in D_n$, and, τ_{-b} denoting the translation of vector $-a$,

$$\tau_{-b} T_{\varepsilon_j,n_j} \to \vec{\mathcal{H}}_L.$$

Step 3: We prove (31) and (32), that is

$$\begin{cases} E_{\varepsilon_j}(u_{\varepsilon_j,n_j}, \check{C}(a, R_0)) &= 2\pi^2\sqrt{1+L^2}|\log \varepsilon_j| + r(\varepsilon_j)|\log \varepsilon_j|, \\ E_{\varepsilon_j}(u_{\varepsilon_j,n_j}, \Omega_{n_j} \setminus \check{C}(a, R_0)) &= 2\pi^2 \log n_j + r(\varepsilon_j)|\log \varepsilon_j|. \end{cases}$$

We first note that from (27) in Corollary 1, the upper bound

$$E_{\varepsilon_j}(u_{\varepsilon_j,n_j}, \check{C}(a, R_0)) \leq 2\pi^2\sqrt{1+L^2}|\log \varepsilon_j| + C_L$$

holds. The lower bound will be a consequence of the isoperimetric type inequality. Indeed, denoting R_{ε_j,n_j} the orthogonal projection of T_{ε_j,n_j} on the plane (x_2, x_3) and $T^1_{\varepsilon_j,n_j}$ the projection on the x_1-axis, we have from claim (202)

$$\mathbf{M}(T_{\varepsilon_j,n_j}) \geq \sqrt{\mathbf{M}(R_{\varepsilon_j,n_j})^2 + \mathbf{M}(T^1_{\varepsilon_j,n_j})^2}.$$

Arguing as for Lemma 6 and using Lemma 5, we deduce

$$\mathbf{M}(T_{\varepsilon_j,n_j}) \geq 2\pi\sqrt{1+L^2} - r(\varepsilon_j).$$

Using *iii)* in Lemma 5, we have the lower bound

$$E_{\varepsilon_j}(u_{\varepsilon_j,n_j}, \check{C}(a, R_0)) \geq 2\pi^2\sqrt{1+L^2}|\log \varepsilon_j| - r(\varepsilon_j)|\log \varepsilon_j|,$$

which finishes the proof of (31). We then infer (32) from (19). □

Step 4: We prove that

$$\frac{d||J_*||}{d\mu_*} = 1 \ \mu_* - a.e., \tag{192}$$

where J_* and μ_* are weak limits (up to another subsequence) of the (translated) jacobian $\tau_{-a}Ju_{\varepsilon_j,n_j}$ and the energy measure $\tau_{-a}\mu_{\varepsilon_j} = e_{\varepsilon_j}(u_{\varepsilon_j,n_j})(a+.)\frac{dx}{|\log\varepsilon_j|}$ **on** $C_{R_0}\cap C_n(-b)$.

In fact, we already know from [JS] and [ABO] that $\frac{d\|J_*\|}{d\mu_*} \leq 1 \ \mu_*$ - a.e.. From Step 1,

$$\tau_{-a}Ju_{\varepsilon_j,n_j} \to \vec{\mathcal{H}}_L \quad \text{ in } [\mathcal{C}^{0,1}_c(\mathbb{T}\times\mathbb{R}^2)]^* \quad \text{ as } j\to+\infty,$$

thus, using Lemma 5 *iii*), we infer

$$\|J_*\| = \mathbf{M}(\mathcal{H}_L) = 2\pi\sqrt{1+L^2}.$$

Moreover, from Step 3, we have

$$\|\mu_*\| \leq 2\pi\sqrt{1+L^2}.$$

Combining these two relations, we are led to the conclusion. □

Step 5: We prove

$$c_{\varepsilon_j,n_j} = \frac{1}{\sqrt{1+L^2}} + r(\varepsilon_j). \tag{193}$$

This relies on the study of the limit equation for the curvature of the singular set given after Theorem 3 in [BOS]. Indeed, applying Theorem 3 in [BOS] for the solution $u_{\varepsilon_j,n_j}(a+.)$ on the domain $C_{R_0}\cap C_n(-b)$, which satisfies the bound (27), we obtain that the varifold $V = V(\Sigma_{\mu_*},\Theta_*)$ satisfies the equation

$$\vec{H}(x) = \star\Big(c\vec{e}_1 \wedge \star\frac{dJ_*}{d\mu_*}\Big),$$

where, we recall, $\vec{H}$ is the generalized mean curvature of V, $\star$ refers to Hodge duality, c is a limit of c_{ε_j,n_j} (bounded sequence in view of (25)), J_* is a weak limit of $\tau_{-a}Ju_{\varepsilon_j,n_j}$, μ_* a weak limit of $\tau_{-a}\mu_{\varepsilon_j,n_j}$ and $\frac{dJ_*}{d\mu_*}$ is the Radon-Nikodym derivative. In fact, it is easy to see that, even though we have a domain depending on j, the equation is valid in the limiting domain (which is the intersection of a cylinder and a half-plane). From Step 4, we know that $\frac{d\|J_*\|}{d\mu_*} = 1 \ \mu_*$-a.e. in Σ_{μ_*}, thus (see Remark 5 in [BOS]) V is a smooth curve and the curvature equation rewrites

$$\vec{\kappa} = c\vec{e}_1 \times \vec{\tau}, \tag{194}$$

where $\vec{\tau}$ is the unit tangent vector and $\vec{\kappa} := \frac{d\vec{\tau}}{ds}$ the curvature vector. From Step 1, the curve is the helix $\vec{\mathcal{H}}_L$, for which $\vec{\tau}(\theta) = (1+L^2)^{-1/2}(1,-L\sin\theta, L\cos\theta)$ and $ds = \sqrt{1+L^2}\ d\theta$, thus

$$\vec{\kappa} = \frac{d\vec{\tau}}{ds} = \frac{d\theta}{ds}\,\frac{d\vec{\tau}}{d\theta} = -\frac{L}{1+L^2}(0,\cos\theta,\sin\theta).$$

Inserting this into (194) yields

$$-\frac{L}{1+L^2}(0,\cos\theta,\sin\theta) = c\vec{e}_1\times\vec{\tau} = -\frac{cL}{\sqrt{1+L^2}}(0,\cos\theta,\sin\theta),$$

from which we deduce the result $c = \frac{1}{\sqrt{1+L^2}}$. □

In view of the uniqueness of the possible limit, we have proved the assertions for all $0<\varepsilon<\varepsilon_0$ sufficiently small and $n \geq C_L|\log\varepsilon|^2$.

8 Proofs of Lemmas 1 and 6

In this Section, we give the proofs of the auxiliary Lemma 1, stating that the vector field $\vec{v}$ behaves like $\frac{\vec{e}_\theta}{r}$ at infinity, and Lemma 6, which exhibits the helix as the unique solution, up to a translation, of an isoperimetric type problem.

8.1 Proof of Lemma 1 : behaviour of $\vec{v}$ at infinity

We recall that Lemma 1 states that $\|\vec{v}-\frac{\vec{e}_\theta}{r}\| \in L^2(\mathbb{T}\times\{r \geq L+1\})$. First, we consider the case $L=0$, for which we denote the vector field $\vec{v}_0$, that is the vortex is the straight line $\mathbb{T}\times\{0\}$. In this case, the Biot-Savart law (11) gives

$$\vec{v}_0 := \frac{1}{2}\int_{-\infty}^{+\infty} \frac{(x-\varphi\vec{e}_1)\times\vec{e}_1}{||x-\varphi\vec{e}_1||^3}\, d\varphi.$$

We will denote (x_1, r, θ) the cylindrical coordinates for x and $(\vec{e}_1, \vec{e}_r, \vec{e}_\theta)$ the corresponding basis, and (x_1, ρ, φ) will be the cylindrical coordinates for $\gamma(\varphi)$ and $(\vec{e}_1, \vec{e}_\rho, \vec{e}_\varphi)$ will be the corresponding basis. Since $x \times \vec{e}_1 = r\vec{e}_\theta$, we have (writing $t = \tan\alpha$, $\alpha \in (-\pi/2, \pi/2)$ for the last integral)

$$\vec{v}_0 = \frac{r\vec{e}_\theta}{2}\int_{-\infty}^{+\infty} \frac{d\varphi}{((x_1-\varphi)^2+r^2)^{3/2}} = \frac{\vec{e}_\theta}{2r}\int_{-\infty}^{+\infty}\frac{dt}{(1+t^2)^{3/2}} = \frac{\vec{e}_\theta}{r}.$$

For the general case, recalling $\gamma(\varphi) = (\varphi, L\cos\varphi, L\sin\varphi) = \varphi\vec{e}_1 + L\vec{e}_\rho$, we first compute

$$\begin{aligned}(x-\gamma(\varphi))\times\gamma'(\varphi) &= (x-\varphi\vec{e}_1)\times\vec{e}_1 + L(x-\varphi\vec{e}_1)\times\vec{e}_\varphi - L\vec{e}_\rho\times\vec{e}_1 - L^2\vec{e}_\rho\times\vec{e}_\varphi \\ &= (x-\varphi\vec{e}_1)\times\vec{e}_1 + Lr\vec{e}_r\times\vec{e}_\varphi - L(x_1-\varphi)\vec{e}_\rho + L\vec{e}_\varphi - L^2\vec{e}_1.\end{aligned}$$

From the Biot-Savart law (11), we deduce

$$\begin{aligned}\vec{v} &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{(x-\gamma(\varphi))\times\gamma'(\varphi)}{||x-\gamma(\varphi)||^3}\, d\varphi \\ &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{(x-\varphi\vec{e}_1)\times\vec{e}_1}{||x-\gamma(\varphi)||^3}\, d\varphi + \frac{Lr}{2}\int_{-\infty}^{+\infty}\frac{\vec{e}_r\times\vec{e}_\varphi}{||x-\gamma(\varphi)||^3}\, d\varphi - \frac{L}{2}\int_{-\infty}^{+\infty}\frac{(x_1-\varphi)\vec{e}_\rho}{||x-\gamma(\varphi)||^3}\, d\varphi \\ &\quad + \frac{L}{2}\int_{-\infty}^{+\infty}\frac{\vec{e}_\varphi}{||x-\gamma(\varphi)||^3}\, d\varphi - \frac{L^2\vec{e}_1}{2}\int_{-\infty}^{+\infty}\frac{d\varphi}{||x-\gamma(\varphi)||^3}.\end{aligned} \tag{195}$$

We then compute, for $r \geq L+1$ and denoting $\lambda := ((x_1-\varphi)^2+r^2)^{1/2} = ||x-\varphi\vec{e}_1||$,

$$||x-\gamma(\varphi)||^{-3} = \lambda^{-3}\Big(1+\mathcal{O}(\lambda^{-1})\Big). \tag{196}$$

Moreover, from the inequality

$$||x-\gamma(\varphi)||^2 \geq \frac{(x_1-\varphi)^2+r^2}{C},$$

valid for $r \geq L+1$ and C depending only on L, we deduce

$$\int_{-\infty}^{+\infty}\frac{d\varphi}{||x-\gamma(\varphi)||^3} \leq \int_{-\infty}^{+\infty}\frac{C\, d\varphi}{((x_1-\varphi)^2+r^2)^{3/2}} = \frac{C}{r^2}\int_{-\infty}^{+\infty}\frac{dt}{(1+t^2)^{3/2}} = \frac{2C}{r^2}, \tag{197}$$

so that the last two integrals in (195) are $\mathcal{O}(r^{-2})$ as $r \to +\infty$. Similarly, $\int_{-\infty}^{+\infty}\lambda^{-4}\, d\varphi \leq Cr^{-3}$. Inserting (196) and (197) into (195) yields for $r \to +\infty$

$$\vec{v} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{(x-\varphi\vec{e}_1)\times\vec{e}_1}{||x-\varphi\vec{e}_1||^3}\, d\varphi + \frac{Lr}{2}\int_{-\infty}^{+\infty}\frac{\vec{e}_r\times\vec{e}_\varphi}{||x-\varphi\vec{e}_1||^3}\, d\varphi - \frac{L}{2}\int_{-\infty}^{+\infty}\frac{(x_1-\varphi)\vec{e}_\rho}{||x-\varphi\vec{e}_1||^3}\, d\varphi + \mathcal{O}(r^{-2}).$$

The first term is $\vec{v}_0 = \frac{\vec{e}_\theta}{r}$. For the first term, we set $\varphi - x_1 = rt$ and obtain

$$\frac{Lr}{2}\int_{-\infty}^{+\infty} \frac{\vec{e}_r \times \vec{e}_\varphi}{||x-\varphi\vec{e}_1||^3}\,d\varphi = \frac{L\vec{e}_r}{2r} \times \int_{-\infty}^{+\infty} \frac{\vec{e}_\varphi(x_1+rt)}{(1+t^2)^{3/2}}\,dt.$$

We then note that $\vec{e}_\varphi(\varphi) = -\frac{d\vec{e}_\rho}{d\varphi}$, thus we may integrate by parts and obtain

$$\frac{Lr}{2}\int_{-\infty}^{+\infty} \frac{\vec{e}_r \times \vec{e}_\varphi}{||x-\varphi\vec{e}_1||^3}\,d\varphi = -\frac{3L\vec{e}_r}{2r^2} \times \int_{-\infty}^{+\infty} \frac{t\vec{e}_\rho(x_1+rt)}{(1+t^2)^{5/2}}\,dt = \mathcal{O}(r^{-2}). \tag{198}$$

The case of the other term is similar. We have therefore proved that, for $r \geq L+1$,

$$\vec{v} = \vec{v}_0 + \mathcal{O}(r^{-2}) = \frac{\vec{e}_\theta}{r} + \mathcal{O}(r^{-2}),$$

which concludes since $r^{-2} \in L^2(\mathbb{T} \times \{r \geq L+1\})$. □

8.2 Proof of Lemma 6 : the isoperimetric type problem

Lemma 6 is the isoperimetric type problem. To prove this Lemma, we proceed as in the proof of Theorem 3.2.27 in [F]. We consider R the orthogonal projection of T on the plane (x_2, x_3) and T^1 on the x_1 axis. Since T has no boundary, neither has R. The current R is therefore compactly supported (since T is), has finite mass (since T has), without boundary and is in $\mathbb{R}^2$, thus, there exists a 2 dimensional integral current S such that

$$R = \partial S$$

(this was not true for T since $\mathbb{T} \times \mathbb{R}^2$ has the homotopy type of the circle: for instance, $\mathbb{T} \times \{0\}$ is not a boundary in $\mathbb{T} \times \mathbb{R}^2$). Choosing S such that (this is possible by [F], theorem 4.2.17)

$$\mathbf{M}(S) = \inf\{\mathbf{M}(S'),\ \partial S' = R\},$$

the following isoperimetric inequality holds (see [A])

$$\mathbf{M}(S) \leq \frac{\mathbf{M}(R)^2}{4\pi}. \tag{199}$$

Moreover, by definition of R and integrating by parts,

$$\mathcal{F}(T) = \pi\langle T, \star\xi\rangle = \pi\langle R, \star\xi\rangle = 2\pi\langle S, \star dx_1\rangle = 2\pi\langle S, dx_2 \wedge dx_3\rangle,$$

that is $\mathcal{F}(T)$ is 2π times the flux of $\vec{e}_1$ through S. Therefore, we have by (199)

$$|\mathcal{F}(T)| \leq 2\pi\mathbf{M}(S) \leq \frac{1}{2}\mathbf{M}(R)^2. \tag{200}$$

Furthermore,

$$|Pr_1(T)| = |\langle T, \star dx_1\rangle| \leq \mathbf{M}(T^1). \tag{201}$$

On the other hand, we claim that

$$\mathbf{M}(T) \geq \sqrt{\mathbf{M}(R)^2 + \mathbf{M}(T^1)^2}. \tag{202}$$

Proof of claim (202)**.** We denote $\sigma : Supp(T) \to \mathbb{N}^*$ the multiplicity of T. Following the proof of Theorem 3.2.27 in [F], we apply Lemma 3.2.25 in [F] to the rectifiable set T: it provides a $\mathcal{H}^1 \llcorner T$-measurable map ξ, with values in the simple 1-vectors of norm 1, such that, for $\mathcal{H}^1 \llcorner T$-a.e. $x \in T$, the subspace associated to $\xi(x)$ is the tangent space to T at x. We decompose $\xi \in \Lambda_1\mathbb{R}^3 \simeq \mathbb{R}^3$ as

$$\xi = \xi_1 + \xi_\top,$$

where $\xi_1 \in \mathbb{R}(1,0,0)$ and $\xi_\top \in Span((0,1,0),(0,0,1)) = (1,0,0)^\perp$. Moreover, by 3.2.20 in [F],

$$\Big|\int_T \sigma\xi_1 \, d\mathcal{H}^1\Big| = \mathbf{M}(T^1) \qquad \text{and} \qquad \int_T \sigma\|\xi_\top\| \, d\mathcal{H}^1 \geq \mathbf{M}(R).$$

Therefore, since $\|\xi\| = 1$ a.e., using Cauchy-Schwarz,

$$\begin{aligned} \mathbf{M}(T)^2 = \mathbf{M}(T)\int_T \sigma\|\xi\|^2 \, d\mathcal{H}^1 &= \mathbf{M}(T)\int_T \sigma|\xi_1|^2 \, d\mathcal{H}^1 + \mathbf{M}(T)\int_T \sigma\|\xi_\top\|^2 \, d\mathcal{H}^1 \\ &\geq \Big|\int_T \sigma\xi_1 \, d\mathcal{H}^1\Big|^2 + \Big(\int_T \sigma\|\xi_\top\| \, d\mathcal{H}^1\Big)^2 \\ &\geq \mathbf{M}(T^1)^2 + \mathbf{M}(R)^2 \end{aligned} \tag{203}$$

and the proof of the claim is complete. □

Combining (200), (201) with the values imposed $\mathcal{F}(T) = 2\pi^2 L^2$ and $Pr_1(T) = 2\pi$, we obtain

$$\mathbf{M}(R)^2 \geq (2\pi L)^2 \qquad \text{and} \qquad \mathbf{M}(T^1) \geq 2\pi,$$

which implies, using (202),

$$2\pi\sqrt{1+L^2} \leq \sqrt{\mathbf{M}(R)^2 + \mathbf{M}(T^1)^2} \leq \mathbf{M}(T),$$

which is the first assertion. If, moreover, we impose $\mathbf{M}(T) \leq 2\pi\sqrt{1+L^2}$, then

$$2\pi\sqrt{1+L^2} \leq \sqrt{\mathbf{M}(R)^2 + \mathbf{M}(T^1)^2} \leq \mathbf{M}(T) \leq 2\pi\sqrt{1+L^2}.$$

Therefore, equality holds everywhere, in particular in (200), (201), (203) and also

$$\mathbf{M}(R) = 2\pi L.$$

We then deduce, with the equality case in the isoperimetric inequality (199), that R is a circle of radius L that is, there exists an $a \in \mathbb{R}^2$ such that (with the natural orientation of $\mathbb{R}^2$ since $\mathcal{F}(T) > 0$)

$$R = \partial D(a, L).$$

We then go back to the equality case in (203) to deduce that σ, ξ_1 and $\|\xi_\top\|$ are constant and, since $Pr_1(T) = 2\pi > 0$, necessarily, $\sigma \equiv 1$, $\xi_1 = c(1,0,0)$ and $\xi_\top = cL(\cos(\theta-\theta_0), \sin(\theta-\theta_0))$, for a $\theta_0 \in \mathbb{R}$ and $c = (1+L^2)^{-1/2}$, thus, there exists a rotation r of axis x_1 and angle θ_0 such that

$$T = (0, a) + r(\vec{\mathcal{H}}_L).$$

Denoting τ the translation of vector $(\theta_0, a) \in \mathbb{T} \times \mathbb{R}^2$, then $T = \tau(\vec{\mathcal{H}}_L)$, which ends the proof. □

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